Published by Institute of Physics Publishing for SISSA
Received: *November 16, 2004*
Accepted: *April 13, 2005*
Published: *May 24, 2005*

Supersymmetric corrections to eleven-dimensional supergravity

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Abstract: In this paper we study eleven-dimensional supergravity in its most general form. This is done by implementing manifest supersymmetry (and Lorentz invariance) through the use of the geometric (torsion and curvature) superspace Bianchi identities. These identities are solved to linear order in a deformation parameter introduced via the dimension zero supertorsion given in its most general form. The theory so obtained is referred to as the deformed theory (to avoid the previously used term "off-shell"). An important by-product of this result is that any higher derivative correction to ordinary supergravity of the same dimension as R^4, but not necessarily containing it, derived e.g. from M-theory, must appear in a form compatible with the equations obtained here. Unfortunately we have not yet much to say about the explicit structure of these corrections in terms of the fields in the massless supermultiplet. Our results are potentially powerful since if the dimension zero torsion could be derived by other means, our reformulation of the Bianchi identities as a number of algebraic relations implies that the full theory would be known to first order in the deformation, including the dynamics. We mention briefly some methods to derive the information needed to obtain explicit answers both in the context of supergravity and ten-dimensional super–Yang-Mills where the situation is better understood. Other relevant aspects like spinorial cohomology, the role of the 3- and 6-form potentials and the connection of these results to M2 and M5 branes are also commented upon.

Keywords: M-Theory, Models of Quantum Gravity, Superspaces, Supergravity Models.

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1. Introduction

Our understanding of M-theory is still very limited, mainly due to the lack of powerful methods to probe it at the microscopic level. One approach to encoding information about M-theory is through its low energy effective field theory. The short distance properties are then built into terms appearing as higher-order corrections to the leading terms given by the action [1]

$$S=\frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left(R-\frac{1}{2\cdot 4!}H^{mnpq}H_{mnpq}\right)+\frac{1}{12\kappa^2}\int C\wedge H\wedge H+ \\ +\text{terms with fermions}\,, \tag{1.1}$$

which is of second order in #(derivatives)+$\frac{1}{2}$#(fermions). The ultimate goal is to be able to derive the higher-derivative corrections, e.g. by means of a microscopic version of M-theory. Since this is not yet possible, our aim here is instead to solve the superspace Bianchi identities in order to obtain the most general form such correction terms can take

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restricted only by supersymmetry and Lorentz invariance in eleven dimensions. To what extent such an approach can capture main features of M-theory is an interesting question to which we have no answer at this point.

The structure of these correction terms is in general extremely complicated. Powers of the Riemann tensor of the kind R^2 and R^4 are basic examples which have been extensively discussed in the literature, primarily in the context of string theory and ten-dimensional effective actions, but also in the eleven-dimensional context relevant to M-theory. A recent overview is given in ref. [2]. The existence of these terms can be inferred by a variety of means in string theory (for a review see [3]), while in M-theory one must rely on anomaly cancellation arguments [4, 5], or (superparticle) loop calculations [6 − 10] in conjunction with results from string theory uplifted to eleven dimensions. Very recently, one-loop calculations were performed directly in eleven dimensions [11], using a generalisation of Berkovits' pure spinor approach [12].

The methods used so far to deduce the explicit form of such corrections in eleven dimensions produce only isolated terms out of a large number of terms making up the complete superinvariants they belong to. For a discussion of superinvariants and a collection of references, see e.g. [13]. Since it would be useful to have a better understanding of the possible superinvariants, there has been a lot of work invested into the supersymmetrisation of some isolated terms. In particular, the supersymmetrisation of R^2 and R^4 terms in ten dimensions were considered already some time ago, see ref. [14] and references therein. More recently also terms related to R^4 in eleven dimensions have been investigated [13, 15] including a detailed study of superinvariants by lifting up results from string vertex operator calculations to eleven dimensions.

Another approach would be to develop methods based on superspace in eleven dimensions [16, 17] that incorporate supersymmetry in a manifest way. In ten dimensions $N = 1$ supergravity has been constructed off-shell at the linear level in terms of a superspace lagrangian [18]. Such a formulation is useful when discussing superinvariants [19, 20], and should in principle lend itself to a complete analysis of possible superinvariants and deduction of the corresponding higher-derivative terms in ordinary component language. The situation in eleven-dimensional supergravity, or M-theory, is, however, completely different due to the fact that an off-shell lagrangian formulation with a finite number of auxiliary fields is not known and may not even exist. From a general counting argument by Siegel and Roček [21] we know that the latter is true for $N = 4$ super–Yang-Mills in four dimensions (and consequently also in ten dimensions) but that maximally supersymmetric supergravity passes the test. Similar arguments [22] suggest that, at the linearised level and in the absence of central charges, eleven-dimensional supergravity does not allow for an off-shell lagrangian quadratic in the fields. The analysis carried out in the present paper will in principle provide an independent check of that statement. In this respect the approach advocated here is parallel to the discussion of ten-dimensional super–Yang-Mills theory carried out in refs. [23] and [24], which proves that an off-shell lagrangian based on these fields does not exist.

In order to implement eleven-dimensional supersymmetry and Lorentz invariance, which must be part of any M-theory effective action, in a manifest way, we will here follow

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refs. [25, 26] and define the theory in superspace by means of the superspace Bianchi identities (SSBI's). The latter are integrability conditions when the theory is formulated in terms of superspace field strengths. By imposing constraints on the supertorsion components of dimension zero the identities turn into non-trivial algebraic relations between certain tensor superfields where some of the relations turn out to be equations of motion. The outcome of the analysis of the SSBI's depends in a crucial way on the choice of the dimensionless components of the supertorsion; setting them equal to a Dirac matrix reproduces uniquely the standard supergravity theory given above in eq. (1.1) as shown by Howe in [27]. The goal in this paper is to complete the analysis of the Bianchi identities started in [26] based on the most general torsion constraints given in [28, 25] and obtain the equations of motion of the deformed theory. Our main result is that we have managed to reformulate the Bianchi identities to first order in a general deformation parameter, into a set of algebraic relations between tensor superfields. These relations act as constraints which any higher derivative correction must satisfy. Unfortunately, we cannot as yet derive explicit expressions for the corrections in terms of the massless physical fields since we have very little information about how to express the dimension zero torsion in terms of these physical fields.

Ultimately, however, we must express the dimension-0 components of the supertorsion in terms of the physical fields. As already mentioned, these components of the supertorsion are not arbitrary, but satisfy certain constraints and may be subjected to certain field redefinitions, as explained in more detail in section 3. The problem of finding an explicit solution to these conditions is equivalent to computing explicit representatives of a particular spinorial cohomology group. This procedure was carried out successfully in the case of $N = 1$, $10d$ SYM in [29 − 32], but the analogous problem in 11d supergravity, where the R^4 terms enter at order ℓ_P^6 (see section 3 for details), seems at present forbiddingly complicated to carry out by brute force. Recently this analysis was carried out to order ℓ_P^3 by one of the authors in [33]. At this order there appears a superinvariant which turns out to be topological in nature in that it can be redefined away by appropriately shifting the flux quantisation condition of the 4-form.

Another approach to finding the explicit form of the torsion constraints, advocated recently in [34], is to use the superspace Bianchi identities for the antisymmetric tensor field. This approach was applied with some success to ten dimensions some time ago [35, 36] in the context of $N = 1$ supergravity coupled to SYM, using in particular the SSBI for the 3-form field strength H with an F^2 topological term $dH = \operatorname{tr} F^2$. In the work presented here the eleven-dimensional 3-form potential emerges from the analysis of the geometric identities via its own field equation and some (possibly anomalous) Bianchi identity, and should not be introduced from the start. However, as pointed out in [34] the superspace Bianchi identities for the 4- and 7-form field strengths do in fact relate the relevant dimension zero torsion components needed in our analysis to perhaps even more basic negative dimension components of the superspace 4-form field strength.[1] The fact that setting all negative dimension components of the 4-form to zero implies via its own SSBI that no torsion deformations are possible, was first noted in [26].

[1] Further comments on the relation between the SSBI for the superspace 4-form and those for the geometric fields, can be found in section 2.7.

It was argued in [34] that, under some plausible assumptions, the SSBI for the 7-form field strength $dH_7 = \frac{1}{2}H_4H_4 + X_8$, where X_8 is the superform extension of the anomaly term found in [5], can be iteratively solved without ambiguities in a similar manner to the ten-dimensional case discussed in [36]. This approach was therefore proposed in [34] as a systematic, albeit quite tedious, way to obtain information about the zero-dimensional torsion components and, eventually, the lagrangian of the deformed theory. In contrast to the situation in eleven dimensions, the problem simplifies enormously in ten dimensions due to the fact that the equations can be solved without relaxing the on-shell torsion constraints [37]. Work on solving the 7-form SSBI in eleven dimensions is in progress [38]. Starting from the torsion SSBI (as in the present paper) and demanding that the anomalous 7-form BI comes out of the analysis at the level of equations of motion at dimension 2, is expected to reflect on the structure of the zero-dimensional torsion components [26].

A proper understanding of the superspace torsion components is also vital when proving κ-invariance for M2 and M5 branes coupled to background supergravity [39 − 41] and M-theory corrected versions of it. In fact, one should compare to the situation in IIA and IIB string theory and the coupling to D-branes [42 − 45]. Here it has been established that there are higher-derivative background field corrections also on the world-volumes of the branes, see e.g. refs. [46, 47] and references therein. World-volume corrections to the M2 brane effective action at order ℓ_P^4 were recently computed directly in eleven dimensions in [48]. The presence of such terms complicates the issue of κ-invariance and it becomes crucial to know the exact form of the supertorsion and to understand its relation to the corrections both in target space and on the brane.

Another aspect of the higher-derivative corrections is that it is to a large extent unclear how supersymmetry organises the infinite set of such terms into infinite subsets unrelated by supersymmetry. From previous work [49] we know, both in ten and eleven dimensions, that adding one bosonic R^2 or R^4 term generates an infinite set of other terms of progressively higher order in the number of derivatives. This is clear in any on-shell theory, as discussed in detail in the type IIB case in e.g. ref. [49]. In the heterotic case in ten dimensions an iterative procedure is needed also due to the fact that there is an implicit dependence on the 3-form field strength in the supercurvature that appears in the SSBI, $dH = \mathrm{tr}(F^2 - R^2)$, used to define the theory in superspace [36, 50]. This situation resembles the one for M-theory under discussion in this paper apart from the fact that the corresponding SSBI for the 4-form field strength is not added as a separate equation but will instead follow from the geometric SSBI for the supertorsion [51, 27].

It is expected that at higher orders there are terms which appear as a result of iteration triggered by a lower-order term, as well as terms which are part of genuinely new superinvariants. Of course, in order to determine which series of terms do actually occur in M-theory, one has to invoke some microscopic description of the theory or rely on a comparison with string theory. In super–Yang-Mills in ten dimensions recent results [52, 53] indicate a situation with new superinvariants appearing at each higher order. Note that if one chooses to truncate the theory to a certain number of fields, as is the case when one considers linearised superinvariants, one generally finds that there are more independent superinvariants (to that order in the number of fields) than when supersymmetry is required to all orders in the number of fields.

This paper is organised as follows. In section two we set up the superspace formalism, review the standard undeformed theory and the issue of the Weyl connection, and discuss how to obtain the torsion constraints and in what sense they will produce the most general deformed theory. The derivation of the deformed theory is then summarised in section three, while the actual details will be spelt out in full detail in appendix B. Section four contains some further comments and conclusions.

2. Superspace formalism and undeformed supergravity

In this section we will review all the relevant formalism and the methods connected to superspace geometry. This is meant to be a self-contained review of superspace geometry with specific application to $11d$ supergravity. We will give a systematic account of superspace, geometrical variables (vielbein, spin connection, torsion, curvature) and differential calculus. We will discuss in detail the issue of torsion constraints, their classification in terms of conventional and physical constraints, their implementation and significance, and show how they affect the Bianchi identities in undeformed $11d$ supergravity. We find it necessary to include this background, since much of the information, especially concerning conventional constraints, is hard to extract from the existing literature and is mostly conveyed as folklore. This will set the stage for deforming the supergravity in the most general way allowed by supersymmetry, which is the main subject of the paper.

2.1 Superspace geometry

The superspace relevant to eleven-dimensional supergravity [1, 16, 17] has 11 bosonic and 32 fermionic directions. The (super-)vielbeins,[2] or frame 1-forms, are $E^A = dZ^M E_M{}^A$, where $M = (m, \mu)$ are coordinate basis ("curved") tangent indices and $A = (a, \alpha)$ are Lorentz frame ("inertial", "flat") tangent indices. Bosonic directions are denoted by Latin letters and fermionic by Greek. $Z^M = (x^m, \theta^\mu)$ are the superspace coordinates.

When dealing with differential forms and exterior derivatives on superspace, we use standard superspace conventions. Since the fermionic property of some components and differentials give signs depending on ordering, this is a convenient way of handling these with a minimum of extra signs. The expansion of a form in components is always done with the differentials in front and in reverse order as compared to the component field:

$$A_{(m)} = \frac{1}{m!} dZ^{M_m} \wedge \cdots \wedge dZ^{M_1} A_{M_1 \ldots M_m} = \frac{1}{m!} E^{A_m} \wedge \cdots \wedge E^{A_1} A_{A_1 \ldots A_n m} \, . \tag{2.1}$$

Taking care of the statistics of the building blocks then means that the components of a wedge product $A_{(m)} \wedge B_{(n)}$ of two bosonic forms come without signs as

$$(A_{(m)} \wedge B_{(n)})_{A_1 \ldots A_{m+n}} = \frac{(m+n)!}{m!n!} A_{[A_1 \ldots A_m} B_{A_{m+1} \ldots A_{m+n})} \tag{2.2}$$

[2] In the following, we will often omit the prefix "super", since it otherwise should be prepended to almost all terms related to superspace geometry.

(or the same expression for the components in coordinate basis), where "$[\cdots)$" indicates graded symmetrisation. By letting the exterior derivative act by wedge product from the right, $dA = A\wedge \overleftarrow{d}$, its component expression is made to mimic the component expansion of the ordinary bosonic exterior derivative, $(dA)_{MM_1\ldots M_n} = (n+1)\partial_{[M}A_{M_1\ldots M_n)}$, which facilitates a translation of identities for bosonic fields.

The superspace is equipped with a spin connection Ω, which is a 1-form taking values in the Lie algebra of the structure group. Working with spin connections (in contrast to affine connections) is a necessary aspect of supergeometry, since the concept of metric is confined to the bosonic directions (for this reason, the term vielbein is misleading; we will use it anyway). The choice of structure group is of course an essential piece of input. We will use the Lorentz group as structure group, with the 32 fermionic components transforming as a spinor, so that $\Omega_{(a}{}^c\eta_{b)c} = 0$, $\Omega_\alpha{}^\beta = \frac{1}{4}(\Gamma^a{}_b)_\alpha{}^\beta\Omega_a{}^b$ and $\Omega_a{}^\beta = 0 = \Omega_\alpha{}^b$. We will later comment on the enlargement of the structure group with a scale transformation, so-called "Weyl superspace" [27]. The choice of the Lorentz group as (a factor in) the structure group is intuitively clear — there must be some input telling the fermions that they are supposed to behave as spinors. We are not aware of any attempt to further modify the structure group, and this is a question to which we hope to be able to devote a systematic investigation in the future.

The spin connection is *a priori* completely unrelated to the vielbein, so the amount of component fields (any field is of course a superfield, depending on all superspace coordinates) is enormous. To take it down to the physical field content of the supergravity theory, one has to make certain choices. Most of those amount to what goes under the name "conventional constraints". Among these are some of a type familiar from the Cartan formulation of ordinary gravity. Finally, a small set of choices, "physical constraints", must be made that have physical significance and determine the exact form of the equations of motion for the supergravity fields. The systematics of the these different types of constraints are explained in detail in the following subsection.

The amount of deviation of the vielbein from being covariantly closed is the torsion, which is a 2-form with an inertial tangent index, $T^A = DE^A = dE^A + E^B\wedge\Omega_B{}^A$ (note that since derivatives act from the right, so does the connection). Torsion is a crucial object in superspace geometry and supergravity, and does not vanish even in flat superspace. Many components will be set to zero by constraints in the following subsection, and the remaining ones constitute, together with curvature, the main tool of our calculations. Curvature is defined as usual, $R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C\wedge\Omega_C{}^B$. Torsion and curvature play the rôle of field strengths in the theory, and obey the Bianchi identities

$$\begin{aligned} DT^A &= E^B\wedge R_B{}^A\,,\\ DR_A{}^B &= 0\,. \end{aligned} \tag{2.3}$$

The first of these plays a central part in the calculations of this paper, while the second need not be explicitly solved since it is implied by the first one. This last fact follows from a theorem by Dragon [54] and relies on the structure group being the Lorentz group.

For completeness, and partially for use in the following subsection, we would like to exhibit the symmetries of the theory. Under the local Lorentz symmetry (or, in general, the structure algebra) with gauge parameter Λ, the connection transforms as $\delta_\Lambda \Omega_A{}^B = D\Lambda_A{}^B$, while the vielbein, torsion and curvature transform covariantly, $\delta_\Lambda E^A = -E^B \wedge \Lambda_B{}^A$, etc. Under diffeomorphisms generated by a vector field $\xi = \xi^M \partial_M = \xi^A E_A$ any field is transformed as $\Delta_\xi \phi = \mathcal{L}_\xi \phi$, where $\mathcal{L}_\xi = i_\xi d + d i_\xi$ is the Lie derivative. In order to covariantise this under the local structure algebra, this transformation is combined with a structure transformation with parameter $-i_\xi \Omega$, so that we instead consider $\tilde{\Delta}_\xi = \mathcal{L}_\xi + \delta_{-i_\xi \Omega}$. The transformation rules of the geometric quantities under consideration are then

$$
\begin{aligned}
\tilde{\Delta}_\xi E^A &= D\xi^A + i_\xi T^A \,, \\
\tilde{\Delta}_\xi \Omega_A{}^B &= i_\xi R_A{}^B \,, \\
\tilde{\Delta}_\xi T^A &= D i_\xi T^A + \xi^B R_B{}^A + E^B \wedge i_\xi R_B{}^A \,, \\
\tilde{\Delta}_\xi R_A{}^B &= D i_\xi R_A{}^B \,.
\end{aligned} \tag{2.4}
$$

All calculations are made with components that carry inertial indices, in order to have access to the structure group and its invariant tensors (gamma matrices). This means that exterior derivatives give rise to torsion,

$$
(DA)_{AA_1\ldots A_n} = (n+1) D_{[A} A_{A_1\ldots A_n)} + \frac{n(n+1)}{2} T_{[AA_1}{}^B A_{|B|A_2\ldots A_n)} \,. \tag{2.5}
$$

We end this section with a comment on the concept of dimension. It is natural to assign to the superspace coordinates canonical (inverse length) dimensions $(-1, -\frac{1}{2})$ (for bosonic and fermionic coordinates, respectively). This introduces a grading. All components of our geometrical objects are conveniently labeled by their canonical dimension. Since an ordinary bosonic vielbein, for example, is dimensionless, the vielbein 1-forms carry dimension -1 (E^a) and $-\frac{1}{2}$ (E^α). Their components have dimension 0 ($E_m{}^a$, $E_\mu{}^\alpha$), $\frac{1}{2}$ ($E_m{}^\alpha$) and $-\frac{1}{2}$ ($E_\mu{}^a$). The torsion has components with dimensions running from 0 to 3/2. For dimensional reasons, only the dimension-0 components ($T_{\alpha\beta}{}^c$) can (and will) contain invariant tensors of the structure group. Any calculation, like the main one of this paper, can be made sequentially for increasing order of dimension, since there are no operators involved that lower the dimension of component fields.

2.2 Conventional constraints

As mentioned in the previous subsection, the constraints we will impose are of different types. The property they have in common is that they are effectuated by fixing some components of the torsion. This ensures the gauge covariance of the constraints, and therefore of the resulting physical system. In principle, some of the constraints have the effect of eliminating certain superfluous components of the vielbein, i.e. components that after solving the SSBI's occur in combinations such that they can be removed by field redefinitions (as can be seen by not enforcing these constraints). However, imposing them explicitly in terms of vielbeins would be unfortunate, since such constraints would break diffeomorphism invariance. The vielbeins carry one coordinate index and one inertial index, and the coordinate index can not be converted into an inertial index (the result would be the unit matrix). The torsion components, on the other hand, carry an inertial index and in addition two lower indices that can be taken in the inertial as well as in the coordinate basis. All constraints are formulated in terms of the torsion, and in terms of components with inertial indices only. Since such components are scalars under diffeomorphisms, this is the only covariant procedure to impose constraints. As long as they are formulated in a way that respects the local structure symmetry, all symmetries will be preserved.

Let us start by considering the *conventional constraints* [55, 56]. There are two kinds of conventional constraints that can be associated with transformations of the spin connection and the vielbein respectively, while the other is held constant. These two transformations have the property that they leave the torsion SSBI in (2.3) invariant and therefore take a solution of the SSBI's into a new solution. This is the reason why we can use these kinds of transformations in order to find an as simple solution to the SSBI's as possible. The two kinds of transformations clearly commute with each other.

The first kind shifts the spin connection by an arbitrary 1-form (with values in the structure algebra) and leaves the vielbein invariant:

$$\left.\begin{aligned} E^A &\to E^A \\ \Omega_A{}^B &\to \Omega_A{}^B + \Delta_A{}^B \end{aligned}\right\} \quad \Longrightarrow \quad T^A \to T^A + E^B \wedge \Delta_B{}^A \,. \tag{2.6}$$

This kind of redefinition serves to remove the independent degrees of freedom in Ω, which can be achieved by constraints on T as long as there are no irreducible representations of the structure group residing in Ω that do not occur in T (all structure groups under consideration fulfill this requirement, as will be seen later). This shift is often expressed as the torsion being absorbed in the spin connection. The canonical example is ordinary bosonic geometry, where one gets $T_{ab}{}^c \to T_{ab}{}^c + 2\Delta_{[ab]}{}^c$, where Δ is antisymmetric in the last two indices, meaning that the transformation can be used to set the torsion identically to zero, leaving the vielbeins as the only independent variables. In supergravity the analysis is more subtle. Only certain representations in the torsion can be brought to zero.

The second kind of transformation consists of a change of tangent bundle, while the connection is left invariant:

$$\left.\begin{aligned} E^A &\to E^B M_B{}^A \\ \Omega_A{}^B &\to \Omega_A{}^B \end{aligned}\right\} \quad \Longrightarrow \quad T^A \to T^B M_B{}^A + E^B \wedge DM_B{}^A \,. \tag{2.7}$$

Again, it is essential that one implements the constraints on the torsion. This will mean that not all components in M can be used. In fact, the remaining degrees of freedom will all reside in the component $E_\mu{}^a$ of negative dimension, as will become clear in section 2.3. The form of the transformation of T will in practice mean that the transformations have to be implemented sequentially in increasing dimension, in order for the second term not to interfere with constraints obtained by using the first term. We will do this in detail for $11d$ supergravity below. This second kind of transformation has no relevance in purely bosonic geometry — there M has dimension 0, and can not be used to algebraically eliminate torsion components of dimension 1 (which are taken care of by the first kind of transformation, anyway). It should also be noted that not all matrices M are relevant. If M is an element in the structure group, the transformations in eq. (2.7) can be supplemented by a transformation of the first kind from eq. (2.6) with suitable parameter ($\Delta = M^{-1}dM + M^{-1}\Omega M - \Omega$) so that the total transformation is a gauge transformation.

2.3 Implementation of the conventional constraints

Having discussed the general aspects of conventional constraints and their associated transformations, we would now like to go through the details for $11d$ supergravity.

The transformations (2.6) and (2.7) act in a highly non-linear way on torsion components with inertial indices. This is because the inertial components even of an invariant differential form change when the frame field is transformed. For example, the first term in the torsion transformation of (2.7) reads

$$T_{AB}{}^{C} \rightarrow (M^{-1})_{A}{}^{A'}(M^{-1})_{B}{}^{B'}T_{A'B'}{}^{C'}M_{C'}{}^{C} + \ldots . \tag{2.8}$$

Instead of considering large transformations, bringing the torsion components in different irreducible representations to their constrained values, we find it much simpler to treat infinitesimal transformations. Then we just have to check that any transformation corresponding to a conventional constraint acts by taking us out of the "constraint surface"; if this is the case, the conventional constraint constitutes a valid choice.

We start by displaying a table of torsion components and transformation parameters (Δ and M), classified according to dimension and further divided into irreducible representations of the Lorentz group.[3]

Dim.	Torsion	Δ	M
$-\frac{1}{2}$			$M_\alpha{}^b$
0	$T_{\alpha\beta}{}^c$ (00000)⊕(01000)⊕(20000) ⊕(10000)⊕(00100)⊕(11000) ⊕(00010)⊕(10002)		$M_a{}^b$ (00000)⊕(01000)⊕(20000)
			$M_\alpha{}^\beta$ (00000)⊕(10000)⊕(01000) ⊕(00100)⊕(00010)⊕(00002)
$\frac{1}{2}$	$T_{\alpha b}{}^c$ (20001)⊕2(10001)⊕(01001) ⊕2(00001)	$\Delta_{\alpha b}{}^c$ (01001)⊕(10001)⊕(00001)	$M_a{}^\beta$ (10001)⊕(00001)
	$T_{\alpha\beta}{}^\gamma$ (00003)⊕(00011)⊕(00101) ⊕2(01001)⊕3(10001)⊕3(00001)		
1	$T_{ab}{}^c$ (11000)⊕(00100)⊕(10000)	$\Delta_{ab}{}^c$ (11000)⊕(00100)⊕(10000)	
	$T_{a\beta}{}^\gamma$		
$\frac{3}{2}$	$T_{ab}{}^\gamma$		

[3]Representations of the Lorentz group Spin(1,10) are specified with standard Dynkin labels, where (10000) is the vector and (00001) the spinor. Note that only the representations relevant for the conventional constraints are explicitly displayed.

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Using a transformation parameter at a certain dimension affects the torsion components at that dimension and higher, so we may implement the conventional constraints sequentially in increasing dimension without the risk of subsequent transformations interfering with conventional constraints already imposed.

At dimension $-1/2$, we have no torsion. Therefore, the transformation with $M_\alpha{}^b$ is not used. This means that we do not remove the degrees of freedom in $E_\mu{}^a$. Note that we want to avoid using a transformation to eliminate degrees of freedom at a higher dimension; this is of course possible in principle, but would not lead to the algebraic elimination of entire superfields.

At dimension 0, it is clear that the torsion components in (11000) and (10002) cannot be algebraically removed, as they do not occur in M. We also note that the transformation $T_{\alpha\beta}{}^c \rightarrow (M^{-1})_\alpha{}^{\alpha'}(M^{-1})_\beta{}^{\beta'}T_{\alpha'\beta'}{}^{c'}M_{c'}{}^c$ is linear in the dimension-0 torsion, so it will not be possible to set it to zero. Starting from the ordinary term $2\Gamma^c_{\alpha\beta}$, it is easily seen that all representations except $(11000) \oplus (10002)$ are generated by a transformation. Out of the representations in the transformation parameter, some are still unused, namely $(00000) \oplus (01000) \oplus (00002)$. The (00000) will be interpreted later as corresponding to a local Weyl (scale) transformation (when supplemented with the suitable transformation of the connection). It is the combination $M_a{}^b = e^\sigma \delta_a{}^b$, $M_\alpha{}^\beta = e^{\sigma/2}\delta_\alpha{}^\beta$ that leaves $\Gamma^a_{\alpha\beta}$ invariant. The (01000) is the combination (infinitesimally) $M_a{}^b = \delta_a{}^b + \varepsilon j_a{}^b$, $M_\alpha{}^\beta = \delta_\alpha{}^\beta + \varepsilon\frac{1}{4}(\Gamma^a{}_b)_\alpha{}^\beta j_a{}^b$ corresponding to a local Lorentz transformation. As argued in the previous subsection, such transformations, lying in the structure group, are irrelevant. In conclusion, the general torsion at dimension 0 is

$$T_{\alpha\beta}{}^c = 2\left(\Gamma_{\alpha\beta}{}^c + \frac{1}{2}\Gamma_{\alpha\beta}{}^{d_1d_2}X_{d_1d_2,}{}^c + \frac{1}{5!}\Gamma_{\alpha\beta}{}^{d_1\ldots d_5}Y_{d_1\ldots d_5,}{}^c\right), \tag{2.9}$$

where X and Y are in the representations (11000) and (10002) of the Lorentz group, respectively, i.e., $X_{[a_1a_2,a]} = 0$, $X_{ab,}{}^b = 0$, $Y_{[a_1\ldots a_5,a]} = 0$, $Y_{a_1\ldots a_4b,}{}^b = 0$.

At dimension 1/2, there is an overlap between the irreducible representations in Δ and M, and one has to check that the corresponding transformations act on T in a non-degenerate way. The choice of which representations to eliminate, among the ones multiply occurring in T, is not unique. Our choice is to eliminate one (10001) and one (00001) representation in each of the $T_{\alpha b}{}^c$, $T_{\alpha\beta}{}^\gamma$.

At dimension 1, finally, the conventional constraints are, as usual, $T_{ab}{}^c = 0$. This part is identical to the elimination of Ω in bosonic gravity.

Once the conventional constraints have been fixed, using the transformations discussed above, certain torsion components are constrained to vanish or to take certain values. The torsion Bianchi identities, which are automatically satisfied when torsion is defined in terms of vielbein and spin connection, then cease to be identities. In $11d$ supergravity, as in other maximally supersymmetric theories lacking an off-shell supersymmetric formulation, the Bianchi identities imply the field equations. The main philosophy of this paper is to turn this property into an advantage. The set of physically distinct theories differ by the choice of non-conventional constraints, as explained in the following subsection. Keeping the torsion components connected to this last choice general does not take the theory off-shell, but gives all allowed forms of the field equations. These components contain fields in a stress tensor multiplet occurring in the field equations.

In conclusion, by using conventional constraints (for the case that the structure group is the Lorentz group), the torsion is brought to the form

$$
\begin{aligned}
\text{dim } 0\text{:}\quad & T_{\alpha\beta}{}^{c} = 2\Big(\Gamma_{\alpha\beta}{}^{c} + \\
& \qquad + \frac{1}{2}\Gamma_{\alpha\beta}{}^{d_1d_2}X_{d_1d_2}{}^{c} + && (11000)\\
& \qquad + \frac{1}{5!}\Gamma_{\alpha\beta}{}^{d_1\ldots d_5}Y_{d_1\ldots d_5}{}^{c}\Big) && (10002)\\
\text{dim } 1/2\text{:}\quad & T_{\alpha b}{}^{c} = \tilde{S}_{b}{}^{c}{}_{\alpha} + && (20001)\\
& \quad + 2(\Gamma_{(b}\tilde{S}_{d)})_{\alpha}\eta^{cd} + && (10001)\\
& \quad + \delta_b^c\tilde{S}_{\alpha} + && (00001)\\
& T_{\alpha\beta}{}^{\gamma} = \frac{1}{120}\Gamma_{\alpha\beta}^{d_1\ldots d_5}\tilde{Z}_{d_1\ldots d_5}{}^{\gamma} + && (00003)\\
& \quad + \frac{1}{24}\Gamma_{\alpha\beta}^{d_1\ldots d_5}(\Gamma_{d_1}\tilde{Z}_{d_2\ldots d_5})^{\gamma} + && (00011)\\
& \quad + \frac{1}{12}\Gamma_{\alpha\beta}^{d_1\ldots d_5}(\Gamma_{d_1d_2}\tilde{Z}_{d_3d_4d_5})^{\gamma} + && (00101)\\
& \quad + \frac{1}{12}\Gamma_{\alpha\beta}^{d_1\ldots d_5}(\Gamma_{d_1d_2d_3}\tilde{Z}_{d_4d_5})^{\gamma} + \frac{1}{2}\Gamma_{\alpha\beta}^{d_1d_2}\tilde{Z}'_{d_1d_2}{}^{c} + && 2(01001)\\
& \quad + \frac{1}{24}\Gamma_{\alpha\beta}^{d_1\ldots d_5}(\Gamma_{d_1\ldots d_4}\tilde{Z}_{d_5})^{\gamma} + \Gamma_{\alpha\beta}^{d_1d_2}(\Gamma_{d_1}\tilde{Z}'_{d_2})^{\gamma} + && 2(10001)\\
& \quad + \frac{1}{120}\Gamma_{\alpha\beta}^{d_1\ldots d_5}(\Gamma_{d_1\ldots d_5}\tilde{Z})^{\gamma} + \frac{1}{2}\Gamma_{\alpha\beta}^{d_1d_2}(\Gamma_{d_1d_2}\tilde{Z}')^{\gamma} && 2(00001)\\
\text{dim } 1\text{:}\quad & T_{ab}{}^{c} = 0\\
& T_{a\beta}{}^{\gamma} = \frac{1}{24}(\Gamma^{d_1\ldots d_4})_{\beta}{}^{\gamma}A_{d_1\ldots d_4a} + \frac{1}{120}(\Gamma_{a}{}^{d_1\ldots d_5})_{\beta}{}^{\gamma}A'_{d_1\ldots d_5} + && 2(00002)\\
& \quad + \frac{1}{6}(\Gamma^{d_1d_2d_3})_{\beta}{}^{\gamma}A_{d_1d_2d_3a} + \frac{1}{24}(\Gamma_{a}{}^{d_1\ldots d_4})_{\beta}{}^{\gamma}A'_{d_1\ldots d_4} + && 2(00010)\\
& \quad + \frac{1}{2}(\Gamma^{d_1d_2})_{\beta}{}^{\gamma}A_{d_1d_2a} + \frac{1}{6}(\Gamma_{a}{}^{d_1d_2d_3})_{\beta}{}^{\gamma}A'_{d_1d_2d_3} + && 2(00100)\\
& \quad + (\Gamma^{d})_{\beta}{}^{\gamma}A_{da} + \frac{1}{2}(\Gamma_{a}{}^{d_1d_2})_{\beta}{}^{\gamma}A'_{d_1d_2} + && 2(01000)\\
& \quad + (\Gamma_{a}{}^{d})_{\beta}{}^{\gamma}A_{d} + A'_{a}\delta_{\beta}{}^{\gamma} + && 2(10000)\\
& \quad + (\Gamma_{a})_{\beta}{}^{\gamma}A + && (00000)\\
& \quad + \frac{1}{120}(\Gamma^{d_1\ldots d_5})_{\beta}{}^{\gamma}B_{d_1\ldots d_5,a} + && (10002)\\
& \quad + \frac{1}{24}(\Gamma^{d_1\ldots d_4})_{\beta}{}^{\gamma}B_{d_1\ldots d_4,a} + && (10010)\\
& \quad + \frac{1}{6}(\Gamma^{d_1d_2d_3})_{\beta}{}^{\gamma}B_{d_1d_2d_3,a} + && (10100)\\
& \quad + \frac{1}{2}(\Gamma^{d_1d_2})_{\beta}{}^{\gamma}B_{d_1d_2,a} + && (11000)
\end{aligned}
$$

$$
\begin{aligned}
&&& + (\Gamma^d)_\beta{}^\gamma B_{d,a} + && (20000) \\
&\text{dim } 3/2: && T_{ab}{}^\gamma = \tilde{t}_{ab}{}^\gamma + && (01001) \\
&&& + 2(\Gamma_{[a}\tilde{t}_{b]})^\gamma + && (10001) \\
&&& + (\Gamma_{ab}\tilde{t})^\gamma && (00001) \qquad (2.10)
\end{aligned}
$$

2.4 Physical (non-conventional) constraints and spinorial cohomology

The form of torsion (2.10) arrived at in the previous subsection is actually the starting point for the calculation of this paper, as it is presented in section 3. It is general enough to contain any "deformation" allowed by supersymmetry, i.e., when substituted in the torsion Bianchi identities it will contain components corresponding to the most general stress tensor multiplet.

In order to arrive at a specific version of $11d$ supergravity, one has to make a few more choices. It was shown in ref. [27] that taking $T_{\alpha\beta}{}^c = 2\Gamma^c_{\alpha\beta}$ at dimension zero gives the superspace formulation of ordinary "undeformed" supergravity. In that paper, the structure group was enlarged to include a Weyl (scale) transformation. As a byproduct of our analysis, we will find the same result for the Lorentz group below.

There exists a very helpful method for determining exactly which torsion components contain information of the deformation, i.e., which torsion components have to be subjected to physical, or non-conventional, constraints, in order to put the theory on-shell expressed in terms of the physical fields. This is the theory of spinorial cohomology, put forward in the context of $10d$ super–Yang-Mills in ref. [29], and further generalised in [31, 57]. A purely tensorial definition, i.e., not relying on particular representations, was given in [34]. We will not give a detailed account of the theory here. Its validity is general and not confined to the supergravity considered in this paper. The statement obtained for $11d$ supergravity is that the gauge transformation (diffeomorphism) parameter ξ^a in (10000), the vielbein $E_\alpha{}^a$ in (10001), the torsion $T_{\alpha\beta}{}^a$ in $(11000)\oplus(10002)$ and the Bianchi identities in $(11001)\oplus(10003)$ are part of a complex

$$
\begin{array}{ccccccccccc}
\xi & \overset{\Delta}{\to} & E & \overset{\Delta}{\to} & T & \overset{\Delta}{\to} & \text{BI} & \overset{\Delta}{\to} & & \cdots \\
(10000) & \to & (10001) & \to & (10002) & \to & (10003) & \to & (10004) & \cdots \\
 & & & \searrow & & \searrow & & \searrow & & \\
 & & & & (11000) & \to & (11001) & \to & (11002) & \cdots \\
 & & & & & & & \searrow & & \\
 & & & & & & & & (12000) & \cdots
\end{array} \qquad (2.11)
$$

The operator Δ is a nilpotent fermionic "exterior derivative" given by the action of the covariant fermionic derivative together with a projection onto the relevant representations. Its cohomology (seen as bosonic/fermionic components of superfields) describes gauge transformations, physical fields and the stress tensor multiplet, respectively (the meaning of cohomology at the level of Bianchi identities and higher has not been understood). The full cohomology of $11d$ supergravity is summarised in the following table, where the entries are denoted by the irreducible representations of the respective component fields, n denotes the horizontal level in the complex (2.11) and the dimensions of the fields are given in the vertical axis.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
dim $=-1$	(10000)						
$-1/2$	(00001)	$\bullet$					
0	$\bullet$	(20000)	$\bullet$				
1/2	$\bullet$	(00001) (10001)	$\bullet$	$\bullet$			
1	$\bullet$	(00010) (10000)	$\bullet$	$\bullet$	$\bullet$		
3/2	$\bullet$	$\bullet$	(00001) (10001)	$\bullet$	$\bullet$	$\bullet$	
2	$\bullet$	$\bullet$	(00000)(00002) (00100)(01000) (10000)(20000)	$\bullet$	$\bullet$	$\bullet$	$\bullet$
5/2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
3	$\bullet$	$\bullet$	$\bullet$	(00000)(00002) (00100)(01000) (10000)(20000)	$\bullet$	$\bullet$	$\bullet$
7/2	$\bullet$	$\bullet$	$\bullet$	(00001) (10001)	$\bullet$	$\bullet$	$\bullet$
4	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(00010) (10000)	$\bullet$	$\bullet$
9/2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(00001) (10001)	$\bullet$	$\bullet$
5	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(20000)	$\bullet$	$\bullet$
11/2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(00001)	$\bullet$
6	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(10000)	$\bullet$
13/2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$

(2.12)

All information is thus contained in the lowest-dimensional superfield of each type. The stress tensor fields, i.e., the deformations, are contained in the torsion representations $(11000)\oplus(10002)$ at dimension 0. These are the ones encoding the exact form of the interactions and, therefore, these are the ones that should be subjected to physical constraints. The undeformed supergravity is thus obtained by imposing the physical constraint

$$T_{\alpha\beta}{}^{c}=2\Gamma^{c}_{\alpha\beta}\,. \tag{2.13}$$

2.5 Bianchi identities and undeformed supergravity

Eleven-dimensional supergravity contains, in addition to the metric and the gravitino, a 3-form potential C with field strength $H=dC$ and field equation $d\star H=\frac{1}{2}H\wedge H$. These fields can be read off the table of spinorial cohomologies at $n=1$, where the tensor field enters via its field strength H, due to gauge invariance. We also find a spinor at dimension $1/2$ and a vector at dimension 1, that will be interpreted as the Weyl connection. Remember that spinorial cohomology is not *a priori* supersymmetric, in that it only encodes objects of lowest dimensionality. Higher-dimensional Bianchi identities will restrict the fields occurring. H may be promoted to a 4-form in superspace, but this is not necessary: like all supergravity fields it is found in the geometric superspace variables. This subsection contains a brief review of the Bianchi identity calculation in the undeformed case, with the purpose of illustrating how the supergravity degrees of freedom arise, how the Bianchi identities lead to the equations of motion, and to what extent the result is unique. Some relevant equations for undeformed supergravity are collected in appendix A.

The torsion Bianchi identity is $DT^A = E^B \wedge R_B{}^A$, which in inertial components reads

$$3R_{[ABC)}{}^D = 3D_{[A}T_{BC)}{}^D + 3T_{[AB}{}^E T_{|E|C)}{}^D \,. \tag{2.14}$$

The procedure for solving the Bianchi identities is to consider this equation, starting from the lowest dimension and moving upwards, decomposing in all occurring irreducible representations of the Lorentz group. If a curvature is allowed to carry a certain representation, the information contained in eq. (2.14) is the value of this curvature component. The only conditions on torsion components come from situations where the curvature is constrained by the structure algebra. The rôle of the structure group is double in this sense: a larger structure group serves on one hand to eliminate torsion components via conventional constraints of the first kind, on the other hand it gives fewer restrictions on the torsion through the Bianchi identities.

The complete set of torsion Bianchi identities is

$$\begin{aligned}
&\text{dim. } 1/2: && 3(\not{R}_{(\alpha\beta})_{\gamma)}{}^d = 3D_{(\alpha}T_{\beta\gamma)}{}^d + 3T_{(\alpha\beta}{}^e T_{|e|\gamma)}{}^d + 3T_{(\alpha\beta}{}^\varepsilon T_{|\varepsilon|\gamma)}{}^d \\
&\text{dim. } 1: && 3(R_{(\alpha\beta})_{\gamma)}{}^\delta = 3D_{(\alpha}T_{\beta\gamma)}{}^\delta + 3T_{(\alpha\beta}{}^e T_{|e|\gamma)}{}^\delta + 3T_{(\alpha\beta}{}^\varepsilon T_{|\varepsilon|\gamma)}{}^\delta \\
& && \updownarrow \\
& && 2(\not{R}_{c(\alpha})_{\beta)}{}^d + (R_{\alpha\beta})_c{}^d = 2D_{(\alpha}T_{\beta)c}{}^d + D_c T_{\alpha\beta}{}^d + T_{\alpha\beta}{}^e \not{T}_{ec}{}^d + T_{\alpha\beta}{}^\varepsilon T_{\varepsilon c}{}^d + \\
& && \qquad + 2T_{c(\alpha}{}^e T_{|e|\beta)}{}^d + 2T_{c(\alpha}{}^\varepsilon T_{|\varepsilon|\beta)}{}^d \\
&\text{dim. } 3/2: && (\not{R}_{\alpha\beta})_c{}^\delta + 2(R_{c(\alpha})_{\beta)}{}^\delta = 2D_{(\alpha}T_{\beta)c}{}^\delta + D_c T_{\alpha\beta}{}^\delta + T_{\alpha\beta}{}^e T_{ec}{}^\delta + T_{\alpha\beta}{}^\varepsilon T_{\varepsilon c}{}^\delta + \\
& && \qquad + 2T_{c(\alpha}{}^e T_{|e|\beta)}{}^\delta + 2T_{c(\alpha}{}^\varepsilon T_{|\varepsilon|\beta)}{}^\delta \\
& && \updownarrow \\
& && (\not{R}_{bc})_\alpha{}^d + 2(R_{\alpha[b})_{c]}{}^d = D_\alpha \not{T}_{bc}{}^d + 2D_{[b}T_{c]\alpha}{}^d + 2T_{\alpha[b}{}^e \not{T}_{|e|c]}{}^d + \not{T}_{bc}{}^e T_{e\alpha}{}^d + \\
& && \qquad + 2T_{\alpha[b}{}^\varepsilon T_{|\varepsilon|c]}{}^d + T_{bc}{}^\varepsilon T_{\varepsilon\alpha}{}^d \\
&\text{dim. } 2: && 2(\not{R}_{\alpha[b})_{c]}{}^\delta + (R_{bc})_\alpha{}^\delta = D_\alpha T_{bc}{}^\delta + 2D_{[b}T_{c]\alpha}{}^\delta + 2T_{\alpha[b}{}^e T_{|e|c]}{}^\delta + \not{T}_{bc}{}^e T_{e\alpha}{}^\delta + \\
& && \qquad + 2T_{\alpha[b}{}^\varepsilon T_{|\varepsilon|c]}{}^\delta + T_{bc}{}^\varepsilon T_{\varepsilon\alpha}{}^\delta \\
& && \updownarrow \\
& && 3(R_{[ab})_{c]}{}^d = 3D_{[a}\not{T}_{bc]}{}^d + 3\not{T}_{[ab}{}^e T_{|e|c]}{}^d + 3T_{[ab}{}^\varepsilon T_{|\varepsilon|c]}{}^d \\
&\text{dim. } 5/2: && 3(\not{R}_{[ab})_{c]}{}^\delta = 3D_{[a}T_{bc]}{}^\delta + 3\not{T}_{[ab}{}^e T_{|e|c]}{}^\delta + 3T_{[ab}{}^\varepsilon T_{|\varepsilon|c]}{}^\delta
\end{aligned} \tag{2.15}$$

Here, we have striked out curvature components that vanish due to the bosonic property of the structure group and torsions that have been set to zero using the ordinary bosonic form of the first kind of conventional constraint, and indicated with arrows curvature components that are related to each other due to the Lorentz condition. Of course both the structure group and the vanishing of certain torsion components has a finer structure than can be taken care of by dividing into bosonic and fermionic indices; it has to be accounted for by performing a full decomposition into irreducible representations. Note that only (linear combinations of) equations without curvature contain information.

According to the previous subsection, the only physical constraint that has to be imposed on the conventionally constrained torsion of eq. (2.10) is

$$T_{\alpha\beta}{}^{c}|_{(11000)\oplus(10002)} = 0\,.$$

The Bianchi identity at dimension 1/2 therefore reads

$$0 = \Gamma^{e}_{(\alpha\beta}T_{|e|\gamma)}{}^{d} + T_{(\alpha\beta}{}^{\varepsilon}\Gamma^{d}_{|\varepsilon|\gamma)}\,. \tag{2.16}$$

Let us compare the content of irreducible representations in this equation, obtained as $(10000)\otimes(00001)^{\otimes_s 3}$, to the one in the torsion according to eq. (2.10).

$$\begin{array}{ll} \text{equation:} & (10003)\oplus(11001)\oplus(20001)\oplus(00003)\oplus(00011)\oplus(00101)\oplus2(01001)\oplus3(10001)\oplus2(00001) \\ \text{torsion:} & \qquad\qquad\qquad\qquad (20001)\oplus(00003)\oplus(00011)\oplus(00101)\oplus2(01001)\oplus3(10001)\oplus3(00001) \end{array} \tag{2.17}$$

From this comparison it follows that the Bianchi identities *may* set the entire dimension-1/2 torsion to 0, except for a spinor. Note that it does not prove that this actually happens; in principle, and this will be the case for Bianchi identities at higher dimension, there can be a linear dependence between equations in the same representation when expressed in terms of the torsion components, leading to more solutions for the torsion than would be guessed by counting representations. At dimension 1/2, however, there is no degeneracy, and everything except for a single spinor is set to zero. A detailed calculation shows that all dimension-1/2 components of the torsion in eq. (2.10) vanish except for the spinors $\tilde{S}$, $\tilde{Z}$ and $\tilde{Z}'$, and that $\tilde{Z} = \frac{3}{88}\tilde{S}$, $\tilde{Z}' = -\frac{1}{44}\tilde{S}$.

The procedure at dimension 1/2 illustrates the general method. At dimension 1, decomposing in irreducible representations and taking into account the Lorentz condition, gives the non-vanishing torsion components A, A_a, A'_a, A_{abcd} and A'_{abcd}, together with the relations (only relations where curvature components are eliminated are displayed)

$$\begin{aligned} D\tilde{S} &= -4224A\,,\\ D\Gamma_a\tilde{S} &= 64A_a = -64A'_a\,,\\ D\Gamma_{ab}\tilde{S} &= 0\,,\\ D\Gamma_{abc}\tilde{S} &= 0\,,\\ D\Gamma_{abcd}\tilde{S} &= -1408(A_{abcd} + 2A'_{abcd})\,,\\ D\Gamma_{abcde}\tilde{S} &= 0\,. \end{aligned} \tag{2.18}$$

Note that this implies $D_{(\alpha}\tilde{S}_{\beta)} = 2\Gamma^{a}_{\alpha\beta}A_a$.

In the following subsection, we demonstrate how the spinor $\tilde{S}$ and the vector A_a are identified as spinor and vector components of a Weyl (scaling) connection, and how they can be brought to zero by a conventional constraint. The rest of the discussion in the present subsection is based on this being done.

The remaining calculation for the undeformed $11d$ supergravity is well known, and we will not relate all details leading to equations of motion etc. In the absence of $\tilde{S}$, one finds the only non-vanishing torsion components at dimension 1 to be A_{abcd} and A'_{abcd}, with the relation $A_{abcd} + 2A'_{abcd} = 0$. This field is identified as proportional to the 4-form field strength H of $11d$ supergravity (see appendix A), i.e., $T_{a\beta}{}^{\gamma} \sim (\Gamma^{d_1d_2d_3})_{\beta}{}^{\gamma}H_{d_1d_2d_3a} -$

$\frac{1}{8}(\Gamma_a{}^{d_1\ldots d_4})_\beta{}^\gamma H_{d_1\ldots d_4}$. At dimension 3/2, the torsion $T_{ab}{}^\gamma$ is the gravitino field strength, and its gamma traces are set to zero as equations of motion, $\tilde{t}_a{}^\gamma = 0$, $\tilde{t}^\gamma = 0$. In addition, one gets from the Bianchi identities information about how the gravitino field strength sits inside the superfield H: $D_\alpha H_{abcd} \sim (\Gamma_{[ab}\tilde{t}_{cd]})^\alpha$. At dimension 2, the Weyl tensor appears in the representation (02000) and is expressible as (schematically) $D\tilde{t} + H^2$. The Bianchi identities at this dimension imply the Bianchi identities as well as the field equations for H together with the Einstein equations.

2.6 The Weyl connection

Apart from the ordinary supergravity fields, the only freedom allowed by the torsion Bianchi identities resides in the spinor superfield $\tilde{S}$ at dimension 1/2. From the dimension-1/2 Bianchi identities it follows that it is constrained to obey the equation $D_{(\alpha}\tilde{S}_{\beta)} - 2\Gamma^c_{\alpha\beta}A_c = 0$. Letting $V_\alpha = \tilde{S}_\alpha$, $V_a = -\frac{1}{2}A_a$, and $V = dZ^A V_A$, this equation is the dimension-1 component of $dV = 0$. Indeed, without going into details, it is confirmed that the Bianchi identities at dimension 3/2 and 2 imply that the 1-form V is closed. Modulo topologically non-trivial configurations, V is exact, $V = d\phi$, where ϕ is a scalar superfield of dimension 0.

We now recall that there was a scalar transformation among the ones connected to conventional constraints that was never used. This "Weyl transformation" can be used to shift ϕ to zero. In the present situation, where we have already chosen conventional constraints at dimension 1/2 and higher, this has to be done carefully. The reason why we always perform conventional transformations by increasing dimension was that they affect torsion components also at higher dimension. Shifting ϕ affects the torsion constraints already fixed, and has to be accompanied by new conventional transformations in order to restore the constraints.[4] For example, at dimension 1/2, all torsion is eliminated by a Weyl rescaling with $\sigma = -\frac{67}{66}\phi$, followed by a conventional transformation of the first kind with $\Delta^c_{\alpha b} = -\frac{1}{66}e^{-\frac{\sigma}{2}}(\Gamma_b{}^c D\phi)_\alpha$ and one of the second kind with $M_a{}^\beta = \frac{1}{132}e^{-\frac{\sigma}{2}}(\Gamma_a D\phi)^\beta$.

If ϕ is a non-trivial flat connection, it can no longer be set to zero, but it can be shifted to a representative in its cohomology class. Such non-trivial Weyl connections have been used to construct massive supergravity in lower dimensions [58, 59]. Even though the formulation due to Howe [27] with a structure group enlarged to encompass scalings is more geometrical, the exact same statements hold true for the Lorentz group formulation: the two versions are completely equivalent (recall the fact that a conventional transformation of the second kind with values in the structure group can be traded for a gauge transformation).

An interesting question is whether the flatness of the Weyl connection remains in the deformed theory. This question can equally well be addressed with or without a Weyl component of the structure group. As we will see in section 3, the answer is negative.

2.7 The 4-form

The 4-form field strength H occurs as a component of the torsion at dimension 1 in the geometric approach to $11d$ supergravity pursued here. As is well known, it can also be

[4] There exists a choice of conventional constraints adapted to Weyl transformations, where this is not needed. Unfortunately, this is not the convention adopted in this paper.

promoted to a 4-form in *superspace*, which we denote by the same letter. Its components have dimensions ranging from -1 ($H_{\alpha\beta\gamma\delta}$) to 1 ($H_{abcd}$). H can be expressed as the exterior derivative of a superspace 3-form potential C, $H = dC$, so its Bianchi identity reads $dH = 0$. Conventional constraints corresponding to redefinitions of the potential may be imposed, analogous to the ones redefining the vielbein in section (2.2), whereupon the Bianchi identities cease to be automatically satisfied.

The gauge transformations, field content and deformations are now related to cohomologies of another complex, namely that containing Γ-traceless parts of n symmetrised spinors:

$$\begin{array}{ccccccccccccc}
 & & \cdots & \overset{\Delta}{\to} & \Lambda & \overset{\Delta}{\to} & C & \overset{\Delta}{\to} & H & \overset{\Delta}{\to} & \mathrm{BI} & \cdots \\
(00000) & \to & (00001) & \to & (00002) & \to & (00003) & \to & (00004) & \to & (00005) & \cdots \\
 & & & \searrow & & \searrow & & \searrow & & \searrow & & \\
 & & & & (01000) & \to & (01001) & \to & (01002) & \to & (01003) & \cdots \\
 & & & & & & & \searrow & & \searrow & & \\
 & & & & & & & & (02000) & \to & (02001) & \cdots
\end{array} \qquad (2.19)$$

The non-conventional constraint that has to be imposed in order to obtain the undeformed supergravity is the vanishing of the dimension -1 components of H in the representations $(02000) \oplus (01002) \oplus (00004)$. The Bianchi identities then imply the equations of motion for the fields.

The cohomology of the complex (2.19) is [31]

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
dim $=-3$	(00000)								
$-5/2$	$\bullet$	$\bullet$							
-2	$\bullet$	(10000)	$\bullet$						
$-3/2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$					
-1	$\bullet$	$\bullet$	(01000) (10000)	$\bullet$	$\bullet$				
$-1/2$	$\bullet$	$\bullet$	(00001)	$\bullet$	$\bullet$	$\bullet$			
0	$\bullet$	$\bullet$	$\bullet$	(00000) (00100) (20000)	$\bullet$	$\bullet$	$\bullet$		
$1/2$	$\bullet$	$\bullet$	$\bullet$	(00001) (10001)	$\bullet$	$\bullet$	$\bullet$	$\bullet$	
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$3/2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(00001) (10001)	$\bullet$	$\bullet$	$\bullet$	$\bullet$
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(00000) (00100) (20000)	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$5/2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(00001)	$\bullet$	$\bullet$	$\bullet$
3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(01000) (10000)	$\bullet$	$\bullet$	$\bullet$
$7/2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
4	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(10000)	$\bullet$	$\bullet$
$9/2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
5	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	(00000)	$\bullet$
$11/2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$

(2.20)

It is interesting to compare the cohomologies here, referred to below as "H-cohomology" to the ones obtained for the geometric quantities stated in eq. (2.12) ("geometric cohomology"). Starting with the gauge transformations, we see that they, in addition to the spinor and vector parameters of superspace diffeomorphisms, contain a 2-form of dimension -1, which is expected. At the level of fields, the 4-form field strength in the geometric cohomology (that can only contain quantities invariant under 2-form gauge transformations) is replaced by the 3-form potential. In addition, there is a scalar at dimension 0. The spinor at dimension 1/2 is still present, but the vector is absent. At the level of the field equations, we find representations fitting the Einstein equations as well as the gravitino equations both in the H-cohomology and in the geometric one. The representation corresponding to the equation of motion for C is present in both, and the Bianchi identity in (00002) has gone away, which is consistent with the formulation being based on the potential instead of the field strength.

In short, the differences between the two cohomologies are in part attributed to the replacement of the field strength by its potential, in part to a difference concerning the Weyl connections.

We should mention that although all fields are contained in the cohomology of the 3-form C, there is no existing formalism based solely on this field, without reference to superspace geometry. One should therefore not *a priori* interpret components of the H-cohomology not present in the geometric cohomology to constitute independent fields or deformations. Similarly, a field or deformation occurring in the geometric cohomology but not in the H-cohomology should not be ruled out by inspection only, since it may be present without explicitly occurring in e.g. the H field.

In the undeformed supergravity, the components in $(02000) \oplus (01002) \oplus (00004)$ at dimension -1 are taken to vanish. This is the non-conventional constraint. The scalar at dimension 0 occurs because $H_{ab\gamma\delta}$ is not invariant under Weyl transformations. It is not possible to set it equal to $2(\Gamma_{ab})_{\gamma\delta}$ by a conventional transformation related to redefinitions of C. A conventional Weyl rescaling of the vielbein is needed for this.

It is clear that the cohomology in $(02000) \oplus (01002) \oplus (00004)$ in $H_{\alpha\beta\gamma\delta}$ is sufficient to encode modifications to the equations of motion for all fields in $11d$ supergravity. A detailed analysis of the superspace Bianchi identities for H up to dimension 0 has been performed in ref. [34].

It has been widely assumed, mostly for æsthetical reasons, that formulations with or without explicit use of H should be equivalent. This is certainly the case for undeformed supergravity. It is not obvious, however, that this statement remains true in the deformed case. As we will see in the following section, the purely geometrical approach of this paper allows for non-vanishing Weyl curvature, which is expressible in terms of the torsion components in $(11000) \oplus (10002)$ at dimension 0. On the other hand, the H-cohomology does not contain the vector component of the Weyl connection. The geometric cohomology contains the Bianchi identity for the H-field, and we are not guaranteed that a deformation will allow for the identification of a globally closed 4-form, although this is of course not excluded.

It was shown in [34] that the system including the H field implies the Bianchi identities in the geometric picture, up to dimension 1/2. Provided no new constraints arise at dimensions higher than 1/2 (this is indeed the case at dimension 1 as we will see in the following sections), this shows that the H field formulation implies the geometric formulation.

For the two formulations to be equivalent the converse should hold as well, and one would expect to find integrability conditions on the X and Y tensors in $(11000) \oplus (10002)$ stating their integrability to the tensors in $(02000) \oplus (01002) \oplus (00004)$. As explained in detail in the following section, so far we have not found any candidates up to dimension 1 for such conditions, other than the constraints in $(11001) \oplus (10003)$. However, it is not at all clear that the latter can play this rôle.

A conclusion concerning the equivalence of our geometric approach with the one containing the superspace 4-form has to await further results at the level of the equations of motion.

3. Deformed supergravity

In this section we solve the SSBI's by using the most general form of the torsion components, subject to the conventional constraints analysed previously. In particular, this implies that the zero-dimension component of the torsion includes the X and Y tensors introduced in equation (2.9). Recall that X and Y are set to zero in the case of ordinary 11d supergravity and as a consequence most of the torsion components are set to zero by the SSBI's. As we have seen in section 2.5, the only components of the torsion that are not set to zero by the SSBI's correspond to the 4-form field strength $H := A_{(4)} = -2A'_{(4)}$ and the gravitino field strength $\tilde{t}$. The curvature tensor R appears at dimension 2.

In the deformed case, this is no longer the case: the SSBI's will now solve for the previously vanishing components of the torsion in terms of (derivatives of) X and Y. It is by substituting X and Y into the SSBI's and solving up to dimension 2, that one arrives at the deformed equations of motion and, eventually, the lagrangian, after specifying X and Y in terms of the physical fields. Clearly, in this approach the deformation is parametrised by X, Y.

Eleven-dimensional supergravity has no coupling constant, since there is no scalar in 11d whose VEV could play this role. There is, however, the possibility of a low-energy (curvature) expansion in the Planck length ℓ_P. It is believed that the first such correction occurs at order ℓ_P^6, corresponding to the still undetermined R^4 superinvariant.[5] As has recently been shown in [33], at order ℓ_P^3 there appears a superinvariant which turns out to be topological in nature in that it can be removed by appropriately shifting the flux quantisation condition of the gauge field. More generally, let us introduce a deformation parameter β and consider the tensors X and Y to be of order β. The reader may want to think of β as being proportional to ℓ_P^6, but our analysis is valid irrespectively of the actual value of β. We treat the problem of solving the deformed Bianchi identities perturbatively, to first order in β. This means, in particular, that we ignore terms quadratic or higher in

[5] Subject to some plausible assumptions, it was argued in [34] that there is a unique R^4 superinvariant consistent with the $C \wedge X_8$ Chern-Simons term.

X, Y.[6] Furthermore the Weyl spinor $\tilde{S}$ is also of order β, since it is set to zero by the SSBI's in the undeformed case. However, as noted in [27], this is only true for a simply connected space-time manifold. We will henceforth assume this to be the case.

Let us now turn to the actual procedure of solving the deformed SSBI's. Just as in the undeformed case, we need to project onto each irreducible representation. This is most conveniently done by appropriately contracting with gamma matrices. The computation is straightforward and conceptually the same as in the undeformed case. It is however much more tedious and we have found GAMMA [61, 62] to be an extremely useful tool.

The reader can find all the details of the calculation in appendix B. Here we summarise a few salient points:

- At dimension 1/2, the BI's impose constraints on the tensors X, Y. Explicitly these read

$$\begin{aligned} Y^1_{a_1\dots a_5,b} &= 0\,, \\ Y^1_{a_1a_2,b} &= \frac{1}{7}X^1_{a_1a_2,b}\,, \end{aligned} \tag{3.1}$$

  where the superscript refers to the θ-level of the corresponding superfield; our notation is further explained in appendix B. These constraints restrict the possible deformations, i.e., the possible admissible expressions of X, Y in terms of physical fields. The problem of finding the explicit form of X, Y which satisfy (3.1) and are not removable by field redefinitions of the form

$$T^a_{\alpha\beta} \to T^a_{\alpha\beta} + D_{(\alpha}\delta E_{\beta)}{}^a\,, \tag{3.2}$$

  is equivalent to solving the spinorial cohomology problem for the theory [31, 57]. The case referred to here, i.e., when X, Y are functions of the physical fields of the theory, was dubbed in [34] 'spinorial cohomology with physical coefficients'. This is to be contrasted with 'spinorial cohomology with unrestricted coefficients', in which case X, Y are freely given superfields. The latter cohomology is summarised, for the case of 11d supergravity, in table (2.12) of section 2.4. Spinorial cohomology with unrestricted coefficients is isomorphic to pure spinor cohomology [24, 63], which has recently found application in the covariant quantisation of the superstring [12].

- All the components of the dimension-1/2 torsion are solved for in terms of (spinor derivatives of) X, Y, except for the Weyl spinor $\tilde{S}$. However, this does not imply that $\tilde{S}$ is an extra degree of freedom, because its derivative $D_{(\alpha}\tilde{S}_{\beta)}$ (which is part of the Weyl curvature) is completely determined in terms X, Y, by the dimension-1 BI. Explicitly

$$\begin{aligned} D\Gamma_a\tilde{S} &= 64A_a\,, \\ \frac{11}{8}D\Gamma_{ab}\tilde{S} &= 4D^fX_{ab,f} + A^{i_1\dots i_4}Y_{i_1\dots i_4[a,b]} + 16A_{ab} + 72A'_{ab}\,, \end{aligned}$$

[6] Note that this an improvement on the analysis of [26, 60] where non-linear terms of the form HY and bosonic derivatives of Y were ignored. In addition, in those references X was set to zero.

$$\frac{11}{8} D\Gamma_{a_1\dots a_5}\tilde{S} = -4D^e Y_{a_1\dots a_5,e} - 120 A_{[a_1a_2a_3}{}^i X_{a_4a_5],i} + \\ + \frac{1}{3}\epsilon_{[a_1a_2a_3|i_1\dots i_8} A^{i_1i_2i_3}{}_{|a_4|} Y^{i_4\dots i_8}{}_{,|a_5]} \,. \tag{3.3}$$

Note that once the deformation is turned on, i.e., for X, $Y \neq 0$, the Weyl curvature ceases to be flat.

- At dimension 1 the SSBI's impose a number of equations which appear to be new constraints on X, Y. Explicitly

$$\begin{aligned}
A \circ X_{a_1a_2,b} &= \frac{11}{15300} A' \circ Y_{a_1a_2,b} + \frac{120}{17} X^2_{a_1a_2,b} - \frac{80}{17} X^{2\prime}_{a_1a_2,b} + \\
&\quad + \frac{280}{17} Y^2_{a_1a_2,b} - \frac{3360}{17} Y^{2\prime}_{a_1a_2,b} && (11000)\,, \\
A \circ Y^{(2)}_{a_1a_2a_3,b} &= \frac{351}{259} A \circ Y^{(1)}_{a_1a_2a_3,b} - \frac{405}{37} DX_{a_1a_2a_3,b} - \frac{1080}{37} X^2_{a_1a_2a_3,b} - \\
&\quad - \frac{1080}{37} X^{2\prime}_{a_1a_2a_3,b} - \frac{1080}{37} Y^2_{a_1a_2a_3,b} - \frac{540}{37} Y^{2\prime}_{a_1a_2a_3,b} - \\
&\quad - \frac{17280}{37} Y^{2\prime\prime}_{a_1a_2a_3,b} && (10100)\,, \\
A \circ X^{(1)}_{a_1\dots a_4,b} &= \frac{81}{37} A \circ X^{(2)}_{a_1\dots a_4,b} + \frac{119}{88800} A \circ Y^{(1)}_{a_1\dots a_4,b} - \frac{217}{66600} A \circ Y^{(2)}_{a_1\dots a_4,b} - \\
&\quad - \frac{7}{111} DY_{a_1\dots a_4,b} + \frac{60}{37} X^2_{a_1\dots a_4,b} + \frac{60}{37} X^{2\prime}_{a_1\dots a_4,b} - \\
&\quad - \frac{30}{37} Y^2_{a_1\dots a_4,b} - \frac{60}{37} Y^{2\prime}_{a_1\dots a_4,b} + \frac{20}{37} Y^{2\prime\prime}_{a_1\dots a_4,b} - \\
&\quad - \frac{80}{37} Y^{2\prime\prime\prime}_{a_1\dots a_4,b} && (10010)\,, \\
A \circ Y^{(2)}_{a_1\dots a_5,b} &= -\frac{11}{2016} A \circ X_{a_1\dots a_5,b} - \frac{11}{42} A \circ Y^{(1)}_{a_1\dots a_5,b} + X^2_{a_1\dots a_5,b} - \\
&\quad - X^{2\prime}_{a_1\dots a_5,b} - \frac{5}{2} Y^2_{a_1\dots a_5,b} - 3Y^{2\prime}_{a_1\dots a_5,b} + \\
&\quad + 5Y^{2\prime\prime}_{a_1\dots a_5,b} + 4Y^{2\prime\prime\prime}_{a_1\dots a_5,b} + \frac{10}{3} Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b} && (10002)\,, \\
DY_{a_1\dots a_5,b} &= \frac{55}{12} A \circ X_{a_1\dots a_5,b} + 3220 A \circ Y^{(1)}_{a_1\dots a_5,b} - 840 X^2_{a_1\dots a_5,b} + \\
&\quad + 840 X^{2\prime}_{a_1\dots a_5,b} - 7980 Y^2_{a_1\dots a_5,b} - 7560 Y^{2\prime}_{a_1\dots a_5,b} + \\
&\quad + 5880 Y^{2\prime\prime}_{a_1\dots a_5,b} + 6720 Y^{2\prime\prime\prime}_{a_1\dots a_5,b} + 5600 Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b} && (10002)\,,
\end{aligned} \tag{3.4}$$

where the quantities involved are defined in appendix B. However, all these should follow from the dimension-1/2 constraints (3.1). This is expected merely on the grounds of representation theory. Namely, taking the tensor product of a spinor and the irreducible representations occurring in the dimension-1/2 constraints, leads to a number of irreducible representations occurring in the dimension-1 SSBI. These are exactly the ones we find above. Explicitly

$$(00001) \otimes (11001) = (11000) \oplus (10100) \oplus (10010) \oplus (10002) \oplus \dots \,,$$

$$(00001) \otimes (10003) = (10002) \oplus \dots . \tag{3.5}$$

In conclusion, no new constraints on X and Y occur at dimension 1.

- The fact that we find no new constraints on X and Y at dimension 1 is a strong indication that the computation we have done is correct,[7] since any computational error generically introduces extra constraints. This also implies that there are no bugs in GAMMA [61, 62].

- Apart from the purely geometrical description of 11d supergravity in terms of the torsion, the system admits an alternative formulation in terms of a closed 4-form in superspace. The SSBI's for the 4-form were analysed in detail in [34] and constraints analogous to (3.1) were derived. It was also shown how to make contact with the supertorsion formulation, by deriving the expressions of X, Y in terms of the lowest, purely spinorial, component of the 4-form. Furthermore it was shown that these expressions should automatically satisfy the supertorsion constraints (3.1). In other words, the 4-form formulation implies the geometric one.

  The converse is less straightforward: If the geometric and the 4-form formulations turn out to be equivalent, the constraints (3.1) will be the integrability conditions for the system to be equivalent to a closed 4-form in superspace. It is far from clear, however, that this will turn out to be the case.

- The problem of computing explicit representatives of spinorial cohomology with physical coefficients is extremely complicated in general, even at order ℓ_P^6. It was argued in [34] that it would be advantageous to tackle this issue within the context of the 4-form (or the 'dual' 7-form) formulation of supergravity. In order to arrive at the deformed equations of motion and eventually at the lagrangian, one would still need to make contact with the geometric formulation of the present paper. In this sense the two approaches are complementary.

- It was argued in [34] that no new constraints appear at dimensions higher than 1/2. We have seen that this is indeed the case at dimension 1. We expect this result to hold at higher dimensions as well. This means in particular that at dimension 3/2 the SSBI's simply solve for the corresponding components of the torsion. In practice, instead of continuing our analysis of the SSBI's at dimension 3/2 or higher, in order to arrive at the equations of motion it is more convenient to simply substitute the explicit expressions of X, Y in terms of the physical fields directly into the BI, along the lines of [30].

Let us briefly review the procedure. As we can see from (2.15) the dimension-3/2 torsion component is given by the spinor derivative of the dimension-1 torsion which is, in its turn, given by two spinor derivatives on X, Y. Schematically:

$$T_{\frac{3}{2}} \sim D_\alpha T_1 \sim D_\alpha^3 X + D_\alpha^3 Y . \tag{3.6}$$

[7]The results in the previous publications [26, 60] are not entirely correct.

The relevant objects to compute are then $D_\alpha X$ and $D_\alpha Y$. To first order in β, this can readily be done as follows. Recall that X, Y are assumed to be functions of the physical field strengths of the theory $H, \tilde{t}, R$. The action of the spinor derivative on the latter is known (from the undeformed theory) to lowest order in β. Schematically

$$\begin{aligned} D_\alpha H &= \tilde{t} + \mathcal{O}(\beta)\,, \\ D_\alpha \tilde{t} &= R + \partial H + H^2 + \mathcal{O}(\beta)\,, \\ D_\alpha R &= \partial \tilde{t} + H\tilde{t} + \mathcal{O}(\beta)\,. \end{aligned} \tag{3.7}$$

As noted before, the tensors X, Y are of order $\mathcal{O}(\beta)$. Therefore the $\mathcal{O}(\beta)$ terms in (3.7) can be ignored as they would give rise to $\mathcal{O}(\beta^2)$ terms in $D_\alpha X$, $D_\alpha Y$.

In appendix B, we use the described method to compute some of the relevant SSBI's at dimension 3/2 and 2, leading to equations of motion.

4. Summary and conclusions

M-theory has, as far as we know, no coupling constants in which to do perturbation theory, and is therefore often viewed as a non-perturbative second quantised theory without well defined one-particle states. As a consequence, in order to avoid discussing the full theory we must rely on some kind of low energy approximation. At low energies the theory has eleven-dimensional supersymmetry and local Lorentz covariance, and one may ask which generalizations of ordinary eleven-dimensional supergravity are compatible with imposing only these symmetries. This may or may not yield a more general structure than a low energy approximation of M-theory.

In this paper we implement these symmetries by the use of superspace. From the supervielbein one defines in a standard fashion the supertorsion and super-Riemann tensors, and derives their respective super-Bianchi identities which we refer to as the geometric SSBI's. This step has in fact introduced all three fields in low energy eleven-dimensional supergravity; the elfbein, the spin 3/2 field and the three-index antisymmetric tensor gauge field. This can be seen by setting the zero dimension torsion tensor equal to a gamma matrix which turns the SSBI's into the lowest order dynamical supergravity equations corresponding to all the three fields [27].

However, as explained in section 2.2, by using only the freedom of performing field redefinitions on the supervielbein and spin connection one finds that the zero dimension torsion component is in the most general situation actually expressed in terms of two unspecified tensor superfields, X and Y, in certain representations of the Lorentz structure group.

In section 3 we have taken a step towards solving the SSBI's in terms of these two tensors by presenting the solution to linear order in X and Y of all SSBI's of dimension 1/2 and 1. This solution is then used in order to obtain deformed equations of motion at dimension 3/2 and 2. The problem of finding explicit forms of the equations of motion is then shifted to finding out the structure of the tensors X and Y in terms of the physical gauge covariant fields, i.e., the Riemann tensor etc. This is a very difficult problem, much

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more complex than the corresponding problem in SYM, simply because the number of independent combinations of fields in the appropriate representations is large, but once the structure of X and Y are known the full theory is obtainable (as can be seen from the formulas in appendix B). The analysis of X and Y is a kind of spinorial cohomology problem, discussed previously in the simpler case of super–Yang-Mills theory in ref. [29, 30, 32]. In the case of supergravity some results were recently derived in ref. [33], where the cohomology was solved to order ℓ_P^3. At this order the first possible non-trivial term appeared which turns out to be purely topological in nature and related to the 4-form quantisation conditions discussed by Witten in ref. [64]. The often discussed R^4 terms are expected at order ℓ_P^6 and will require a substantial amount of work to analyse in full generality. Even the task of just writing down an Ansatz for X and Y in terms of the physical fields (obeying the undeformed field equations) looks formidable, since the independent combinations in the representations of X and Y at this dimension are counted in thousands. We would like to return to this in a future publication, but think that input of some other kind is needed to avoid that type of brute force calculation.

A different approach to finding the form of X and Y is to introduce also the superspace BI's for the gauge fields, either the one for the 4-form field strength only or in combination with the SSBI's for the dual 6-form potential. In the latter case the anomaly related term C_3X_8 in the lagrangian can be most naturally introduced in superspace via the generalised SSBI $dH_7 = \frac{1}{2}H_4 \wedge H_4 + X_8$. Once this is done the central role played by the dimension zero torsion is taken over by the lowest dimension component of the the 4-form H_4, namely $H_{\alpha\beta\gamma\delta}$ as discussed in detail in [34]. Restricting this field affects the structure of the theory, e.g., setting it to zero leaves the theory in the lowest second order form [26]. More important, however, is that the deformation in the geometric sector can probably more easily be derived by relating it to the deformation in the gauge sector, as emphasised in [34, 38].

In fact, the geometric sector of the superspace version of the theory may be viewed as secondary to the gauge sector. That is, since the supertorsion tensor appears explicitly in the component equations of the gauge SSBI's one can obtain the geometric deformation in terms of the deformation in the gauge sector and consistency will probably also require the geometric SSBI's to be satisfied. The cohomology tables presented in section 2 have a bearing on this issue. As one can see by comparing the tables, the geometric and gauge systems do not seem to be in a one to one correspondence, a fact that is not yet understood. The differences were discussed in section 2.7, where some of them where explained. Some differences remain obscure, however, among them the question of the existence of a closed 4-form in the geometric formulation. Perhaps the most efficient way to proceed will in the end turn out to be to use all the SSBI's simultaneously, provided the discrepancies between the cohomology tables are not a symptom of any deeper structural differences between the two systems.

Acknowledgments

We would like to thank P. Howe and P. Vanhove for useful discussions. This work is supported in part by the Swedish Research Council.

A. Undeformed 11d supergravity in superspace

The nonzero components of the supertorsion and supercurvature of undeformed 11d supergravity are given by

$$
\begin{aligned}
T_{\alpha\beta}{}^{c} &= 2(\Gamma^{c})_{\alpha\beta} \\
T_{a\beta}{}^{\gamma} &= -\frac{1}{36}\left((\Gamma^{bcd})_{\beta}{}^{\gamma}H_{abcd} + \frac{1}{8}(\Gamma_{abcde})_{\beta}{}^{\gamma}H^{abcd}\right)
\end{aligned}
\tag{A.1}
$$

(the field strength H is related to the torsion component A used in this paper by $H = -6A$) and

$$
\begin{aligned}
R_{\alpha\beta ab} &= -\frac{1}{3}\left((\Gamma^{cd})_{\alpha\beta}H_{abcd} + \frac{1}{24}(\Gamma_{abcdef})_{\alpha\beta}H^{cdef}\right) \\
R_{\alpha bcd} &= -(\Gamma_{b}T_{cd})_{\alpha} - (\Gamma_{c}T_{bd})_{\alpha} + (\Gamma_{d}T_{bc})_{\alpha}\,.
\end{aligned}
\tag{A.2}
$$

Note that the Lorentz condition implies

$$
R_{AB\alpha}{}^{\beta} = \frac{1}{4}R_{ABcd}(\Gamma^{cd})_{\alpha}{}^{\beta}\,. \tag{A.3}
$$

The action of the spinorial derivative on the physical field strengths and their equations of motion are given by

$$
\begin{aligned}
D_{\alpha}H_{abcd} &= -12(\Gamma_{[ab}T_{cd]})_{\alpha} \\
D_{\alpha}T_{ab}{}^{\beta} &= \frac{1}{4}R_{abcd}(\Gamma^{cd})_{\alpha}{}^{\beta} - 2D_{[a}T_{b]\alpha}{}^{\beta} - 2T_{[a|\alpha}{}^{\epsilon}T_{|b]\epsilon}{}^{\beta} \\
D_{\alpha}R_{abcd} &= 2D_{[a|}R_{\alpha|b]cd} - T_{ab}{}^{\epsilon}R_{\epsilon\alpha cd} + 2T_{[a|\alpha}{}^{\epsilon}R_{\epsilon|b]cd}
\end{aligned}
\tag{A.4}
$$

and

$$
\begin{aligned}
D_{[a}H_{bcde]} &= 0 \\
D^{f}H_{fabc} &= -\frac{1}{2(4!)^{2}}\varepsilon_{abcd_1\ldots d_8}H^{d_1\ldots d_4}H^{d_5\ldots d_8} \\
(\Gamma^{a}T_{ab})_{\alpha} &= 0 \\
R_{ab} - \frac{1}{2}\eta_{ab}R &= -\frac{1}{12}\left(H_{adfg}H_{b}{}^{dfg} - \frac{1}{8}\eta_{ab}H_{dfge}H^{dfge}\right)\,.
\end{aligned}
\tag{A.5}
$$

The above equations can be integrated to an action whose bosonic part is given by eq. (1.1).

B. Solution of the SSBI's

In this section we present the full details of our solution to the SSBI's at dimension 1/2 and dimension 1, and partial results (deformations of the equations of motion) at dimension 3/2 and 2. The procedure was explained in the main body of the paper: the torsion components are expanded in irreducible representations as in (2.10) of section 2.5, and substituted into the SSBI's given in equation (2.15). By appropriately contracting with gamma matrices one then projects onto each irreducible representation. The fields $H_{abcd} := A_{abcd} = -2A'_{abcd}$, $\tilde{t}_{ab}{}^{\gamma}$, R_{abcd}, are the only ones that are non-zero in the ordinary (undeformed) eleven-dimensional supergravity. All other superfields are of linear order in the deformation parameter β, as explained in section 3. In the following we discard terms of order $\mathcal{O}(\beta^2)$. This is the *only* approximation we use.

A note on notation: a tilde denotes a spinor superfield (e.g. $\tilde{S}$). A numerical superscript n on the superfield A denotes a superfield A^n sitting at θ^n-level in A.

B.1 The dimension-1/2 SSBI's

The SSBI at dimension 1/2 reads

$$0 = R_{(\alpha\beta\gamma)}{}^{d} = D_{(\alpha}T_{\beta\gamma)}{}^{d} + T_{(\alpha\beta}{}^{E}T_{|E|\gamma)}{}^{d}\,. \tag{B.1}$$

It decomposes in irreducible representations as

$$\begin{aligned}(00001)^{\otimes_s 3} \otimes (10000) = 2 \times (00001) \oplus (00003) \oplus (00011) \oplus (00101) \oplus \\ \oplus\, 2 \times (01001) \oplus 3 \times (10001) \oplus (10003) \oplus (11001) \oplus (20001)\end{aligned} \tag{B.2}$$

Since the SSBI involves the fields at θ^1-level in X and Y, we also need to expand $D_\alpha X_{a_1a_2,b}$ and $D_\alpha Y_{a_1\ldots a_5,b}$ into irreducible tensors. Explicitly:

$$\begin{aligned}DY_{a_1\ldots a_5,b} = {} & 5\left(\Gamma_{[a_1}Y^1_{a_2\ldots a_5]b} + \Gamma_b Y^1_{a_1\ldots a_5}\right) + \\ & + 10\left(\Gamma_{[a_1a_2}Y^1_{a_3a_4a_5]b} + \Gamma_{b[a_1}Y^1_{a_2\ldots a_5]} + \frac{6}{7}Y^1_{[a_1\ldots a_4}\eta_{a_5]b}\right) + \\ & + 10\left(\Gamma_{[a_1a_2a_3}Y^1_{a_4a_5]b} + \Gamma_{b[a_1a_2}Y^1_{a_3a_4a_5]} + \frac{12}{7}\eta_{b[a_1}\Gamma_{a_2}Y^1_{a_3a_4a_5]}\right) + \\ & + 5\left(\Gamma_{[a_1\ldots a_4}Y^1_{a_5]b} + \Gamma_{b[a_1a_2a_3}Y^1_{a_4a_5]} + \frac{18}{7}\eta_{b[a_1}\Gamma_{a_2a_3}Y^1_{a_4a_5]}\right) + \\ & + \left(\Gamma_{a_1\ldots a_5}Y^1_b + \Gamma_{b[a_1\ldots a_4}Y^1_{a_5]} + \frac{24}{7}\eta_{b[a_1}\Gamma_{a_2a_3a_4}Y^1_{a_5]}\right) + \\ & + Y^1_{a_1\ldots a_5,b} + 5\Gamma_{[a_1}Y^1_{a_2\ldots a_5],b} + 10\Gamma_{[a_1a_2}Y^1_{a_3a_4a_5],b} + \\ & + 10\Gamma_{[a_1a_2a_3}Y^1_{a_4a_5],b} + 5\Gamma_{[a_1\ldots a_4}Y^1_{a_5],b}\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}DX_{a_1a_2,b} = {} & 2\left(\Gamma_b X^1_{a_1a_2} - \Gamma_{[a_1}X^1_{a_2]b}\right) - \left(\Gamma_{a_1a_2}X^1_b - \Gamma_{b[a_1}X^1_{a_2]} + \frac{3}{10}X^1_{[a_1}\eta_{a_2]b}\right) \\ & + 2\Gamma_{[a_1}X^1_{a_2],b} + X^1_{a_1a_2,b}\,.\end{aligned} \tag{B.4}$$

The inversions which we will need later are

$$\begin{aligned}X^1_a &= \frac{10}{1287}\Gamma^{ij}DX_{ij,a}\,, \\ X^1_{a_1a_2} &= \frac{4}{117}(\Gamma^i DX_{a_1a_2,i} - \frac{1}{8}\Gamma_{[a_1|}{}^{ij}DX_{ij,|a_2]})\,,\end{aligned}$$

$$
\begin{aligned}
X^1_{a_1,a_2} &= \frac{10}{117}\big(\Gamma^i DX_{i(a_1,a_2)} + \frac{3}{20}\Gamma_{(a_1|}{}^{ij} DX_{ij|,a_2)}\big)\,,\\
&\dots\,,
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
Y^1_{a_1\dots a_5} &= \frac{1}{195}\bigg(\Gamma^i DY_{a_1\dots a_5,i} - \Gamma_{[a_1}{}^{ij} DY_{a_2\dots a_5]i,j} - \frac{1}{2}\Gamma_{[a_1a_2}{}^{ijk} DY_{a_3a_4a_5]ij,k} +\\
&\qquad + \frac{1}{6}\Gamma_{[a_1a_2a_3}{}^{ijkl} DY_{a_4a_5]ijk,l} + \frac{1}{24}\Gamma_{[a_1\dots a_4}{}^{ijklm} DY_{a_5]ijkl,m}\bigg)\,,\\
Y^1_{a_1\dots a_4} &= \frac{1}{312}\bigg(\Gamma^{ij} DY_{a_1\dots a_4 i,j} + \frac{4}{5}\Gamma_{[a_1}{}^{ijk} DY_{a_2a_3a_4]ij,k} - \frac{3}{10}\Gamma_{[a_1a_2}{}^{ijkl} DY_{a_3a_4]ijk,l} -\\
&\qquad - \frac{1}{15}\Gamma_{[a_1a_2a_3}{}^{ijklm} DY_{a_4]ijkl,m}\bigg)\,,\\
Y^1_{a_1a_2a_3} &= -\frac{1}{936}\bigg(\Gamma^{ijk} DY_{a_1a_2a_3ij,k} - \frac{3}{5}\Gamma_{[a_1}{}^{ijkl} DY_{a_2a_3]ijk,l} - \frac{3}{20}\Gamma_{[a_1a_2}{}^{ijklm} DY_{a_3]ijkl,m}\bigg)\,,\\
Y^1_{a_1a_2} &= -\frac{1}{3510}\bigg(\Gamma^{ijkl} DY_{a_1a_2ijk,l} + \frac{2}{5}\Gamma_{[a_1}{}^{ijklm} DY_{a_2]ijkl,m}\bigg)\,,\\
Y^1_a &= \frac{1}{61776}\Gamma^{ijklm} DY_{ijklm,a}\,,\\
Y^1_{a_1,a_2} &= \frac{1}{5616}\bigg(\Gamma^{ijkl} DY_{ijkl(a_1,a_2)} - \frac{6}{35}\Gamma_{(a_1|}{}^{ijklm} DY_{ijklm,|a_2)}\bigg)\,,\\
&\dots\,,
\end{aligned}
\tag{B.6}
$$

where the ellipses stand for the irreducible representations which drop out of the SSBI's and will not be needed in the following. Plugging the above and the explicit expressions for the torsion components (2.10) into the SSBI, we get

$$\tilde{Z}_{a1\dots a5} = -5Y^1_{a_1\dots a_5}\,, \tag{B.7}$$

$$\tilde{Z}_{a1\dots a4} = -\frac{13}{7}Y^1_{a_1\dots a_4}\,, \tag{B.8}$$

$$\tilde{Z}_{a1\dots a3} = -\frac{39}{14}Y^1_{a_1\dots a_3}\,, \tag{B.9}$$

$$\tilde{Z}_{ab} = \frac{1}{8}X^1_{ab} - \frac{255}{112}Y^1_{ab}\,, \tag{B.10}$$

$$\tilde{Z}'_{ab} = -\frac{17}{8}X^1_{ab} + \frac{25}{16}Y^1_{ab}\,, \tag{B.11}$$

$$\tilde{S}_a = \frac{693}{460}X^1_a - \frac{198}{115}Y^1_a\,, \tag{B.12}$$

$$\tilde{Z}_a = \frac{221}{920}X^1_a - \frac{741}{1610}Y^1_a\,, \tag{B.13}$$

$$\tilde{Z}'_a = -\frac{923}{460}X^1_a + \frac{221}{115}Y^1_a\,, \tag{B.14}$$

$$\tilde{Z} = \frac{3}{88}\tilde{S}\,, \tag{B.15}$$

$$\tilde{Z}' = -\frac{1}{44}\tilde{S}\,, \tag{B.16}$$

$$Y^1_{a_1\dots a_5,b} = 0\,, \tag{B.17}$$
$$Y^1_{a1a2,b} = \frac{1}{7}X^1_{a_1a_2,b}\,, \tag{B.18}$$
$$\tilde{S}_{a,b} = X^1_{a,b} + 14Y^1_{a,b}\,. \tag{B.19}$$

B.2 The dimension-1 SSBI's

We now turn to the SSBI's at dimension 1. There are two such equations, namely

$$R_{\alpha\beta c}{}^{d} = 2D_{(\alpha}T_{\beta)c}{}^{d} + D_cT_{\alpha\beta}{}^{d} + T_{\alpha\beta}{}^{E}T_{Ec}{}^{d} + 2T_{c(\alpha}{}^{E}T_{|E|\beta)}{}^{d}\,, \tag{B.20}$$
$$R_{(\alpha\beta\gamma)}{}^{\delta} = D_{(\alpha}T_{\beta\gamma)}{}^{\delta} + T_{(\alpha\beta}{}^{E}T_{|E|\gamma)}{}^{\delta}\,. \tag{B.21}$$

These decompose as

$$\begin{aligned}(00001)^{\otimes_s 2}\otimes(10000)^{\otimes 2} = {}&(00000)\oplus 3\times(10000)\oplus 3\times(01000)\oplus\\ &\oplus 2\times(00100)\oplus 2\times(00010)\oplus 3\times(00002)\oplus\\ &\oplus 2\times(10002)\oplus 2\times(10010)\oplus 2\times(10100)\oplus\\ &\oplus 2\times(11000)\oplus 2\times(20000)\oplus\dots\end{aligned} \tag{B.22}$$

and

$$\begin{aligned}(00001)^{\otimes_s 3}\otimes(00001) = {}&(00000)\oplus 2\times(10000)\oplus 3\times(01000)\oplus\\ &\oplus 3\times(00100)\oplus 3\times(00010)\oplus 4\times(00002)\oplus\\ &\oplus 3\times(10002)\oplus 2\times(10010)\oplus 2\times(10100)\oplus\\ &\oplus 2\times(11000)\oplus(20000)\oplus\dots\,,\end{aligned} \tag{B.23}$$

respectively. We will need the θ^2-level expansion of $X_{ab,c}$, $Y_{abcde,f}$. We have

$$\begin{aligned}(00001)^{\otimes_a 2}\otimes(10002) = {}&(01000)\oplus 2\times(00100)\oplus 2\times(00010)\oplus 2\times(00002)\\ &\oplus 5\times(10002)\oplus 4\times(10010)\oplus 3\times(10100)\oplus\\ &\oplus 2\times(11000)\oplus\times(20000)\oplus\dots\end{aligned} \tag{B.24}$$

and

$$\begin{aligned}(00001)^{\otimes_a 2}\otimes(11000) = {}&(01000)\oplus(00100)\oplus(00010)\oplus(00002)\\ &\oplus 2\times(10002)\oplus 2\times(10010)\oplus 2\times(10100)\oplus\\ &\oplus 2\times(11000)\oplus\times(20000)\oplus\dots\,.\end{aligned} \tag{B.25}$$

Explicitly we expand

$$\begin{aligned}&\frac{1}{10}D_{[\alpha}D_{\beta]}Y_{a_1\dots a_5,b} =\\ &\quad = \Gamma_{[a_1a_2}{}^{e}Y^2_{a_3a_4a_5]be} + \Gamma_{b[a_1}{}^{e}Y^2_{a_2\dots a_5]e} + \frac{6}{7}\eta_{b[a_1}\Gamma_{a_2}{}^{e_1e_2}Y^2_{a_3a_4a_5]e_1e_2} +\\ &\qquad +\frac{1}{6}\left(\Gamma_{[a_1\dots a_4}{}^{e_1e_2e_3}Y^{2\prime}_{a_5]be_1e_2e_3} + \Gamma_{b[a_1a_2a_3}{}^{e_1e_2e_3}Y^{2\prime}_{a_4a_5]e_1e_2e_3} + \frac{6}{7}\eta_{b[a_1}\Gamma_{a_2a_3a_4}{}^{e_1\dots e_4}Y^{2\prime}_{a_5]e_1\dots e_4}\right) +\end{aligned}$$

$$
\begin{aligned}
&+\Gamma_{[a_1a_2a_3}{}^{e}Y^{2}_{a_4a_5]be}+\Gamma_{b[a_1a_2}{}^{e}Y^{2}_{a_3a_4a_5]e}-\frac{6}{7}\eta_{b[a_1}\Gamma_{a_2a_3}{}^{e_1e_2}Y^{2}_{a_4a_5]e_1e_2}+\\
&+\frac{1}{6}\left(\Gamma_{a_1\dots a_5}{}^{e_1e_2e_3}Y^{2\prime}_{be_1e_2e_3}+\Gamma_{b[a_1\dots a_4}{}^{e_1e_2e_3}Y^{2\prime}_{a_5]e_1e_2e_3}-\frac{6}{7}\eta_{b[a_1}\Gamma_{a_2\dots a_5]}{}^{e_1\dots e_4}Y^{2\prime}_{e_1\dots e_4}\right)+\\
&+\Gamma_{[a_1a_2a_3}Y^{2}_{a_4a_5]b}+\Gamma_{b[a_1a_2}Y^{2}_{a_3a_4a_5]}-\frac{6}{7}\eta_{b[a_1}\Gamma_{a_2a_3}{}^{e}Y^{2}_{a_4a_5]e}+\\
&+\frac{1}{2}\left(\Gamma_{a_1\dots a_5}{}^{e_1e_2}Y^{2\prime}_{be_1e_2}+\Gamma_{b[a_1\dots a_4}{}^{e_1e_2}Y^{2\prime}_{a_5]e_1e_2}-\frac{6}{7}\eta_{b[a_1}\Gamma_{a_2\dots a_5]}{}^{e_1e_2e_3}Y^{2\prime}_{e_1e_2e_3}\right)+\\
&+\Gamma_{[a_1\dots a_4}Y^{2}_{a_5]b}+\Gamma_{b[a_1a_2a_3}Y^{2}_{a_4a_5]}+\frac{6}{7}\eta_{b[a_1}\Gamma_{a_2a_3a_4}{}^{e}Y^{2}_{a_5]e}+\\
&+\frac{1}{2}\left(\Gamma_{[a_1a_2|}{}^{e_1e_2}Y^{2}_{e_1e_2|a_3a_4a_5],b}+\Gamma_{b[a_1|}{}^{e_1e_2}Y^{2}_{e_1e_2|a_2\dots,a_5]}\right)+\\
&+\frac{1}{2}\left(\Gamma_{[a_1a_2|}{}^{e_1e_2}Y^{2\prime}_{e_1e_2|a_3a_4a_5],b}+\Gamma_{[a_1a_2|}{}^{e_1e_2}Y^{2\prime}_{e_1e_2b|a_3a_4,a_5]}+\frac{4}{7}\eta_{b[a_1}\Gamma_{a_2|}{}^{e_1e_2e_3}Y^{2\prime}_{e_1e_2e_3|a_3a_4,a_5]}\right)+\\
&+\frac{1}{24}\left(\Gamma_{[a_1\dots a_4|}{}^{e_1\dots e_4}Y^{2\prime\prime}_{e_1\dots e_4|a_5],b}+\Gamma_{b[a_1a_2a_3|}{}^{e_1\dots e_4}Y^{2\prime\prime}_{e_1\dots e_4|a_4,a_5]}\right)+\\
&+\frac{1}{24}\left(\Gamma_{[a_1\dots a_4|}{}^{e_1\dots e_4}Y^{2\prime\prime\prime}_{e_1\dots e_4|a_5],b}+\Gamma_{[a_1\dots a_4|}{}^{e_1\dots e_4}Y^{2\prime\prime\prime}_{e_1\dots e_4b,|a_5]}+\right.\\
&\qquad\left.+\frac{24}{35}\eta_{b[a_1}\Gamma_{a_2a_3a_4|}{}^{e_1\dots e_5}Y^{2\prime\prime\prime}_{e_1\dots e_5,|a_5]}\right)+\\
&+Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b}+\\
&+\Gamma_{[a_1a_2|}{}^{e}Y^{2}_{e|a_3a_4a_5],b}+\Gamma_{b[a_1|}{}^{e}Y^{2}_{e|a_2\dots,a_5]}-\frac{3}{14}\eta_{b[a_1}\Gamma_{a_2|}{}^{e_1e_2}Y^{2}_{e_1e_2|a_3a_4,a_5]}+\\
&+\Gamma_{[a_1a_2|}{}^{e}Y^{2\prime}_{e|a_3a_4a_5],b}+\Gamma_{[a_1a_2|}{}^{e}Y^{2\prime}_{eb|a_3a_4,a_5]}-\frac{5}{7}\eta_{b[a_1}\Gamma_{a_2|}{}^{e_1e_2}Y^{2\prime}_{e_1e_2|a_3a_4,a_5]}+\\
&+\frac{1}{6}\left(\Gamma_{[a_1\dots a_4|}{}^{e_1e_2e_3}Y^{2\prime\prime}_{e_1e_2e_3|a_5],b}+\Gamma_{b[a_1a_2a_3|}{}^{e_1e_2e_3}Y^{2\prime\prime}_{e_1e_2e_3|a_4,a_5]}-\right.\\
&\qquad\left.-\frac{1}{28}\eta_{b[a_1}\Gamma_{a_2a_3a_4|}{}^{e_1\dots e_4}Y^{2\prime\prime}_{e_1\dots e_4,|a_5]}\right)+\\
&+\frac{1}{6}\left(\Gamma_{[a_1\dots a_4|}{}^{e_1e_2e_3}Y^{2\prime\prime\prime}_{e_1e_2e_3|a_5],b}+\Gamma_{[a_1\dots a_4|}{}^{e_1e_2e_3}Y^{2\prime\prime\prime}_{e_1e_2e_3b,|a_5]}-\frac{5}{7}\eta_{b[a_1}\Gamma_{a_2a_3a_4|}{}^{e_1\dots e_4}Y^{2\prime\prime\prime}_{e_1\dots e_4,|a_5]}\right)+\\
&+\Gamma_{[a_1a_2a_3|}{}^{e}Y^{2}_{e|a_4a_5],b}+\Gamma_{b[a_1a_2|}{}^{e}Y^{2}_{e|a_3a_4,a_5]}-\frac{2}{7}\eta_{b[a_1}\Gamma_{a_2a_3|}{}^{e_1e_2}Y^{2}_{e_1e_2|a_4,a_5]}+\\
&+\Gamma_{[a_1a_2a_3|}{}^{e}Y^{2\prime}_{e|a_4a_5],b}-\Gamma_{[a_1a_2a_3|}{}^{e}Y^{2\prime}_{eb|a_4,a_5]}-\frac{6}{7}\eta_{b[a_1}\Gamma_{a_2a_3|}{}^{e_1e_2}Y^{2\prime}_{e_1e_2|a_4,a_5]}+\\
&+\frac{1}{6}\left(\Gamma_{a_1\dots a_5}{}^{e_1e_2e_3}Y^{2\prime\prime}_{e_1e_2e_3,b}+\Gamma_{b[a_1\dots a_4|}{}^{e_1e_2e_3}Y^{2\prime\prime}_{e_1e_2e_3,|a_5]}\right)+\\
&+\Gamma_{[a_1a_2a_3}Y^{2}_{a_4a_5],b}+\frac{6}{7}\eta_{b[a_1}\Gamma_{a_2a_3|}{}^{e}Y^{2}_{e|a_4,a_5]}+\\
&+\frac{1}{2}\left(\Gamma_{a_1\dots a_5}{}^{e_1e_2}Y^{2\prime}_{e_1e_2,b}+\Gamma_{b[a_1\dots a_4|}{}^{e_1e_2}Y^{2\prime}_{e_1e_2,|a_5]}\right)+\\
&+\Gamma_{[a_1a_2a_3a_4}Y^{2}_{a_5],b}+\frac{4}{7}\eta_{b[a_1}\Gamma_{a_2a_3a_4|}{}^{e}Y^{2}_{e,|a_5]}+\\
&+\cdots
\end{aligned}
\tag{B.26}
$$

and

$$
\begin{aligned}
&\frac{1}{2}D_{[\alpha}D_{\beta]}X_{a_1a_2,b}=\\
&\qquad=\frac{1}{6}\left(\Gamma_{[a_1}{}^{e_1e_2e_3}X^{2}_{a_2]be_1e_2e_3}-\Gamma_{b}{}^{e_1e_2e_3}X^{2}_{a_1a_2e_1e_2e_3}-\frac{3}{10}\eta_{b[a_1}\Gamma^{e_1\dots e_4}X^{2}_{a_2]e_1\dots e_4}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\Gamma_{[a_1}{}^{e_1e_2}X^2_{a_2]be_1e_2}-\Gamma_b{}^{e_1e_2}X^2_{a_1a_2e_1e_2}-\frac{3}{10}\eta_{b[a_1}\Gamma^{e_1e_2e_3}X^2_{a_2]e_1e_2e_3}\right)+\\
&+\frac{1}{2}\left(\Gamma_{a_1a_2}{}^{e_1e_2}X^2_{be_1e_2}-\Gamma_{b[a_1}{}^{e_1e_2}X^2_{a_2]e_1e_2}+\frac{3}{10}\eta_{b[a_1}\Gamma_{a_2]}{}^{e_1e_2e_3}X^2_{e_1e_2e_3}\right)+\\
&+\Gamma_{a_1a_2}{}^{e}X^2_{be}-\Gamma_{b[a_1}{}^{e}X^2_{a_2]e}+\frac{3}{10}\eta_{b[a_1}\Gamma_{a_2]}{}^{e_1e_2}X^2_{e_1e_2}+\\
&+\frac{1}{6}(\Gamma^{e_1e_2e_3}X^2_{e_1e_2e_3a_1a_2,b}-\Gamma^{e_1e_2e_3}X^2_{e_1e_2e_3b[a_1,a_2]})+\\
&+\frac{1}{120}(\Gamma_{a_1a_2}{}^{e_1\dots e_5}X^{2\prime}_{e_1\dots e_5,b}-\Gamma_{b[a_1}{}^{e_1\dots e_5}X^{2\prime}_{e_1\dots e_5,|a_2]})+\\
&+\frac{1}{6}\left(\Gamma_{[a_1|}{}^{e_1e_2e_3}X^2_{e_1e_2e_3|a_2],b}-\Gamma_b{}^{e_1e_2e_3}X^2_{e_1e_2e_3[a_1,a_2]}-\frac{1}{20}\eta_{b[a_1|}\Gamma^{e_1\dots e_4}X^2_{e_1\dots e_4,|a_2]}\right)+\\
&+\frac{1}{6}\left(\Gamma_{[a_1|}{}^{e_1e_2e_3}X^{2\prime}_{e_1e_2e_3|a_2],b}+\Gamma_{[a_1|}{}^{e_1e_2e_3}X^{2\prime}_{e_1e_2e_3b,|a_2]}+\frac{1}{8}\eta_{b[a_1|}\Gamma^{e_1\dots e_4}X^{2\prime}_{e_1\dots e_4,|a_2]}\right)+\\
&+\frac{1}{2}\left(\Gamma_{[a_1|}{}^{e_1e_2}X^2_{e_1e_2|a_2],b}-\Gamma_b{}^{e_1e_2}X^2_{e_1e_2[a_1,a_2]}+\frac{1}{30}\eta_{b[a_1|}\Gamma^{e_1e_2e_3}X^2_{e_1e_2e_3,|a_2]}\right)+\\
&+\frac{1}{2}\left(\Gamma_{[a_1|}{}^{e_1e_2}X^{2\prime}_{e_1e_2|a_2],b}+\Gamma_{[a_1|}{}^{e_1e_2}X^{2\prime}_{e_1e_2b,|a_2]}-\frac{2}{15}\eta_{b[a_1|}\Gamma^{e_1e_2e_3}X^{2\prime}_{e_1e_2e_3,|a_2]}\right)+\\
&+\frac{1}{2}(\Gamma_{a_1a_2}{}^{e_1e_2}X^2_{e_1e_2,b}-\Gamma_{b[a_1|}{}^{e_1e_2}X^2_{e_1e_2,|a_2]})+\\
&+X^{2\prime}_{a_1a_2,b}+\\
&+\Gamma_{a_1a_2}{}^{e}X^2_{e,b}-\Gamma_{b[a_1|}{}^{e}X^2_{e,|a_2]}+\\
&+\cdots\,.
\end{aligned}
\tag{B.27}
$$

Let us also note that

$$
\begin{aligned}
2D_{(\alpha}D_{\beta)}X_{a_1a_2,b}=&\,2\left[2(\Gamma^{ij})_{\alpha\beta}A_{ij[a_1|}{}^{c}-\frac{1}{6}(\Gamma_{[a_1|}{}^{cijkl})_{\alpha\beta}A'_{ijkl}\right]X_{c|a_2],b}+\\
&+\left[2(\Gamma^{ij})_{\alpha\beta}A_{ijb}{}^{c}-\frac{1}{6}(\Gamma_b{}^{cijkl})_{\alpha\beta}A'_{ijkl}\right]X_{a_1a_2,c}-\\
&-2(\Gamma^f)_{\alpha\beta}D_fX_{a_1a_2,b}
\end{aligned}
\tag{B.28}
$$

and

$$
\begin{aligned}
2D_{(\alpha}D_{\beta)}Y_{a_1\dots a_5,b}=&\,5\left[2(\Gamma^{ij})_{\alpha\beta}A_{ij[a_1|}{}^{c}-\frac{1}{6}(\Gamma_{[a_1|}{}^{cijkl})_{\alpha\beta}A'_{ijkl}\right]Y_{c|a_2\dots a_5],b}+\\
&+\left[2(\Gamma^{ij})_{\alpha\beta}A_{ijb}{}^{c}-\frac{1}{6}(\Gamma_b{}^{cijkl})_{\alpha\beta}A'_{ijkl}\right]Y_{a_1\dots a_5,c}-\\
&-2(\Gamma^f)_{\alpha\beta}D_fY_{a_1\dots a_5,b}\,.
\end{aligned}
\tag{B.29}
$$

We are now ready to project onto each irreducible representation.

The singlet. From the 1st SSBI we get

$$
(\Gamma^{bc})^{\alpha\beta}R_{\alpha\beta bc}=-14080A\,.
\tag{B.30}
$$

From the 2nd SSBI we get

$$3(\Gamma_e)^{\alpha\beta}(\Gamma^e)_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta = \frac{7}{2}(\Gamma^{bc})^{\alpha\beta}R_{\alpha\beta bc}$$
$$= 9856A + 14D^\alpha S_\alpha\,. \tag{B.31}$$

Eqs. (B.30), (B.31) give,

$$D\tilde{S} = -4224A\,. \tag{B.32}$$

The vectors. From the 1st SSBI we get

$$(\Gamma^b)^{\alpha\beta}R_{\alpha\beta ba} = 2D\Gamma_a\tilde{S} + 1280A_a + 128A'_a\,, \tag{B.33}$$
$$(\Gamma^c)^{\alpha\beta}R_{\alpha\beta ac} = 2D\Gamma_a\tilde{S} + 1408A'_a\,, \tag{B.34}$$
$$0 = \delta^{bc}(\Gamma_a)^{\alpha\beta}R_{\alpha\beta bc} = 22D\Gamma_a\tilde{S} - 1280A_a + 128A'_a\,. \tag{B.35}$$

From the 2nd SSBI we get

$$3(\Gamma^e)^{\alpha\beta}(\Gamma_{ea})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta = 8(\Gamma^b)^{\alpha\beta}R_{\alpha\beta ab}$$
$$= 20D\Gamma_a\tilde{S} - 10240A_a + 1280A'_a\,, \tag{B.36}$$
$$3(\Gamma_a)^{\alpha\beta}\delta_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta = (\Gamma^b)^{\alpha\beta}R_{\alpha\beta ab}$$
$$= 34D\Gamma_a\tilde{S} - 1280A_a + 2176A'_a\,. \tag{B.37}$$

From eqs. (B.33)–(B.37) we get

$$\begin{aligned} D\Gamma_a\tilde{S} &= 64A_a\,,\\ A'_a &= -A_a\,. \end{aligned} \tag{B.38}$$

The 2-forms. From the 1st SSBI we get

$$\begin{aligned}(\Gamma_{[a_1|}{}^b)^{\alpha\beta}R_{\alpha\beta b|a_2]} = {}& 2D\Gamma_{a_1a_2}\tilde{S} - \frac{960}{23}D^eX_{a_1a_2,e} -\\ & - \frac{209664}{115}X^2_{a_1a_2} + \frac{29952}{23}Y^2_{a_1a_2} -\\ & - \frac{64}{3}A^{i_1\ldots i_4}Y_{i_1\ldots i_4[a_1,a_2]} + \frac{2288}{69}A'^{i_1\ldots i_4}Y_{i_1\ldots i_4[a_1,a_2]} -\\ & - 128A_{a_1a_2} + 1152A'_{a_1a_2}\,,\end{aligned} \tag{B.39}$$

$$\begin{aligned}(\Gamma_{[a_1|}{}^c)^{\alpha\beta}R_{\alpha\beta|a_2]c} = {}& 2D\Gamma_{a_1a_2}\tilde{S} - \frac{224}{23}D^eX_{a_1a_2,e} -\\ & - \frac{209664}{115}X^2_{a_1a_2} + \frac{29952}{23}Y^2_{a_1a_2} -\\ & - \frac{96}{23}A'^{i_1\ldots i_4}Y_{i_1\ldots i_4[a_1,a_2]} - 1280A_{a_1a_2}\,,\end{aligned} \tag{B.40}$$

$$\begin{aligned}0 = \delta^{bc}(\Gamma_{a_1a_2})^{\alpha\beta}R_{\alpha\beta bc} = {}& 22D\Gamma_{a1a2}\tilde{S} - 64D^eX_{a_1a_2,e} -\\ & - \frac{32}{3}A^{i_1\ldots i_4}Y_{i_1\ldots i_4[a_1,a_2]} + \frac{32}{3}A'^{i_1\ldots i_4}Y_{i_1\ldots i_4[a_1,a_2]} -\end{aligned}$$

$$-256A_{a_1a_2}-1152A'_{a_1a_2}\,. \tag{B.41}$$

From the 2nd SSBI we get

$$\begin{aligned}
3(\Gamma_{a_1a_2})^{\alpha\beta}\delta_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& -2(\Gamma_{[a_1|}{}^b)^{\alpha\beta}R_{\alpha\beta b|a_2]}\\
=&\frac{382}{11}D\Gamma_{a_1a_2}\tilde{S}-\frac{349696}{3289}D^eX_{a_1a_2,e}-\\
&-\frac{5359104}{1265}X^2_{a_1a_2}+\frac{1552896}{253}Y^2_{a_1a_2}-\\
&-\frac{32}{3}A^{i_1\dots i_4}Y_{i_1\dots i_4[a_1,a_2]}+\frac{118688}{9867}A'^{i_1\dots i_4}Y_{i_1\dots i_4[a_1,a_2]}-\\
&-256A_{a_1a_2}-1152A'_{a_1a_2}\,,
\end{aligned} \tag{B.42}$$

$$\begin{aligned}
3(\Gamma_{[a_1})^{\alpha\beta}(\Gamma_{a_2]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =&\,0\\
=&\frac{14}{11}D\Gamma_{a_1a_2}\tilde{S}+\frac{3072}{253}D^eX_{a_1a_2,e}+\\
&+\frac{2875392}{1265}X^2_{a_1a_2}-\frac{1198080}{253}Y^2_{a_1a_2}+\\
&+\frac{32}{3}A^{i_1\dots i_4}Y_{i_1\dots i_4[a_1,a_2]}-\frac{3200}{759}A'^{i_1\dots i_4}Y_{i_1\dots i_4[a_1,a_2]}-\\
&-2048A_{a_1a_2}-1152A'_{a_1a_2}\,,
\end{aligned} \tag{B.43}$$

$$\begin{aligned}
3(\Gamma^e)^{\alpha\beta}(\Gamma_{ea_1a_2})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =&\,14(\Gamma_{[a_1|}{}^b)^{\alpha\beta}R_{\alpha\beta b|a_2]}\\
=&\frac{126}{11}D\Gamma_{a_1a_2}\tilde{S}+\frac{398848}{3289}D^eX_{a_1a_2,e}+\\
&+\frac{10639872}{253}X^2_{a_1a_2}-\frac{14685696}{253}Y^2_{a_1a_2}+\\
&+32A^{i_1\dots i_4}Y_{i_1\dots i_4[a_1,a_2]}+\frac{13536}{3289}A'^{i_1\dots i_4}Y_{i_1\dots i_4[a_1,a_2]}-\\
&-2304A_{a_1a_2}-10368A'_{a_1a_2}\,.
\end{aligned} \tag{B.44}$$

From eqs. (B.39)–(B.44) we get

$$\begin{aligned}
\frac{11}{8}D\Gamma_{ab}\tilde{S} =&\,4D^fX_{ab,f}+A^{i_1\dots i_4}Y_{i_1\dots i_4[a,b]}+16A_{ab}+72A'_{ab}\,,\\
A_{ab} =& -\frac{1}{320528}(1636D^fX_{ab,f}+1289A^{i_1\dots i_4}Y_{i_1\dots i_4[a,b]})-\\
&-\frac{18}{7705}(87X^2_{ab}+365Y^2_{ab})\,,\\
A'_{ab} =&\,\frac{11}{480792}(1056D^fX_{ab,f}+881A^{i_1\dots i_4}Y_{i_1\dots i_4[a,b]})+\\
&+\frac{2}{1541}(1907X^2_{ab}-2131Y^2_{ab})\,,\\
A^{i_1\dots i_4}Y_{i_1\dots i_4[a,b]} =& -2A'^{i_1\dots i_4}Y_{i_1\dots i_4[a,b]}\,.
\end{aligned} \tag{B.45}$$

Note that the last line is redundant, as it is implied by the zeroth order equation $A_{i_1\dots i_4}=-2A'_{i_1\dots i_4}$.

The 3-forms. From the 1st SSBI we get

$$
\begin{aligned}
(\Gamma_{a_1a_2a_3}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc} = & -\frac{2}{5}\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} + 192A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]} + \\
& + \frac{8}{15}\epsilon_{[a_1a_2|i_1\dots i_9}A'^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} - 1152A'^{ij}{}_{[a_1a_2|}X_{ij,|a_3]} - \\
& - 7168A_{a_1a_2a_3}\,,
\end{aligned} \tag{B.46}
$$

$$
\begin{aligned}
(\Gamma_{[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2a_3]} = & -\frac{2}{45}\epsilon_{[a_1a_2|i_1\dots i_9}A'^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} - 64A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]} - \\
& - 128A_{a_1a_2a_3}\,.
\end{aligned} \tag{B.47}
$$

From the 2nd SSBI we get

$$
\begin{aligned}
3(\Gamma_{[a_1})^{\alpha\beta}(\Gamma_{a_2a_3]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta = & -15(\Gamma_{[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2a_3]} + \frac{1}{2}(\Gamma_{a_1a_2a_3}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc} \\
= & \, 2D\Gamma_{a_1a_2a_3}\tilde{S} - \frac{139776}{23}X^2_{a_1a_2a_3} - \frac{146432}{23}Y^2_{a_1a_2a_3} - \\
& - \frac{1597440}{23}Y^{2\prime}_{a_1a_2a_3} - 1920A_{a_1a_2a_3} + 1024A'_{a_1a_2a_3} + \\
& + \frac{2}{15}\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} + \frac{576}{23}A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]} + \\
& + \frac{2}{69}\epsilon_{[a_1a_2|i_1\dots i_9}A'^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} + 192A'^{ij}{}_{[a_1a_2|}X_{ij,|a_3]}\,,
\end{aligned} \tag{B.48}
$$

$$
\begin{aligned}
3(\Gamma_{[a_1a_2})^{\alpha\beta}(\Gamma_{a_3]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta = & \, (\Gamma_{[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2a_3]} + \frac{1}{2}(\Gamma_{a_1a_2a_3}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc} \\
= & \, \frac{6}{11}D\Gamma_{a_1a_2a_3}\tilde{S} + \frac{6404608}{1265}X^2_{a_1a_2a_3} - \frac{611328}{253}Y^2_{a_1a_2a_3} + \\
& + \frac{3522560}{253}Y^{2\prime}_{a_1a_2a_3} + 128A_{a_1a_2a_3} - 1024A'_{a_1a_2a_3} - \\
& - \frac{2}{15}\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} + \frac{797504}{3289}A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]} + \\
& + \frac{21634}{148005}\epsilon_{[a_1a_2|i_1\dots i_9}A'^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} + 192A'^{ij}{}_{[a_1a_2|}X_{ij,|a_3]}\,,
\end{aligned} \tag{B.49}
$$

$$
\begin{aligned}
3(\Gamma^e)^{\alpha\beta}(\Gamma_{ea_1a_2a_3})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta = & -24(\Gamma_{[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2a_3]} + 2(\Gamma_{a_1a_2a_3}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc} \\
= & \, \frac{80}{11}D\Gamma_{a_1a_2a_3}\tilde{S} + \frac{23052288}{1265}X^2_{a_1a_2a_3} - \frac{6889472}{253}Y^2_{a_1a_2a_3} - \\
& - \frac{14008320}{253}Y^{2\prime}_{a_1a_2a_3} - 3072A_{a_1a_2a_3} + 10240A'_{a_1a_2a_3} + \\
& + \frac{4}{15}\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} - \frac{3891072}{3289}A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]} - \\
& - \frac{6496}{49335}\epsilon_{[a_1a_2|i_1\dots i_9}A'^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} - 768A'^{ij}{}_{[a_1a_2|}X_{ij,|a_3]}\,.
\end{aligned} \tag{B.50}
$$

A useful identity is

$$
\epsilon_{a_1a_2a_3i_1\dots i_8}A^{i_1i_2i_3j}Y_j{}^{i_4\dots i_7,i_8} = -\frac{3}{20}\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]}\,. \tag{B.51}
$$

From eqs. (B.46)–(B.50) we get,

$$
\begin{aligned}
A'_{a_1a_2a_3} =& A_{a_1a_2a_3}\,,\\
A_{a_1a_2a_3} =& -\frac{5457}{66976}A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]}+\frac{193}{4592640}\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]}-\\
&-\frac{1}{230}(593X^2_{a_1a_2a_3}-60Y^2_{a_1a_2a_3}+2900Y^{2\prime}_{a_1a_2a_3})\,,\\
D\Gamma_{a_1a_2a_3}\tilde{S} =& \frac{1546236}{2093}A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]}-\frac{14671}{35880}\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]}+\\
&+\frac{64}{115}(7239X^2_{a_1a_2a_3}+5540Y^2_{a_1a_2a_3}+71100Y^{2\prime}_{a_1a_2a_3})\,,
\end{aligned}
\tag{B.52}
$$

by imposing the conditions

$$
\begin{aligned}
A^{ij}{}_{[a_1a_2|}X_{ij,|a_3]} &= -2A'^{ij}{}_{[a_1a_2|}X_{ij,|a_3]}\\
\epsilon_{[a_1a_2|i_1\dots i_9}A^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]} &= -2\epsilon_{[a_1a_2|i_1\dots i_9}A'^{i_1\dots i_4}Y^{i_5\dots i_9}{}_{,|a_3]}\,,
\end{aligned}
$$

which are implied by the zeroth-order equation $A_{a_1\dots a_4}=-2A'_{a_1\dots a_4}$.

The 4-forms. From the 1st SSBI we get

$$
\begin{aligned}
(\Gamma_{a_1a_2a_3a_4}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc} =& -512A^{ijk}{}_{[a_1|}Y_{ijk|a_2a_3,a_4]}+1024A'^{ijk}{}_{[a_1|}Y_{ijk|a_2a_3,a_4]}-\\
&-5376A'_{a_1a_2a_3a_4}\,,
\end{aligned}
\tag{B.53}
$$

$$
\begin{aligned}
(\Gamma_{[a_1a_2|})^{\alpha\beta}R_{\alpha\beta|a_3a_4]} =& \frac{64}{3}A^{ijk}{}_{[a_1|}Y_{ijk|a_2a_3,a_4]}-\frac{128}{3}A'^{ijk}{}_{[a_1|}Y_{ijk|a_2a_3,a_4]}+\\
&+128A_{a_1a_2a_3a_4}\,.
\end{aligned}
\tag{B.54}
$$

From the 2nd SSBI we get

$$
\begin{aligned}
3(\Gamma_{[a_1})^{\alpha\beta}(\Gamma_{a_2a_3a_4]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& \frac{1}{2}(\Gamma_{a_1a_2a_3a_4}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc}\\
=& \frac{2080512}{1265}X^2_{a_1a_2a_3a_4}+\frac{1400320}{253}Y^2_{a_1a_2a_3a_4}+\\
&+\frac{6266880}{253}Y^{2\prime}{}_{a_1a_2a_3a_4}+\frac{774016}{3289}A^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}-\\
&-\frac{660480}{3289}A'^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}+\frac{14}{11}D\Gamma_{a_1a_2a_3a_4}\tilde{S}+\\
&+1792A_{a_1a_2a_3a_4}+896A'_{a_1a_2a_3a_4}\,,
\end{aligned}
\tag{B.55}
$$

$$
\begin{aligned}
3(\Gamma_{[a_1a_2})^{\alpha\beta}(\Gamma_{a_3a_4]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& -14(\Gamma_{[a_1a_2|})^{\alpha\beta}R_{\alpha\beta|a_3a_4]}+\frac{1}{2}(\Gamma_{a_1a_2a_3a_4}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc}\\
=& \frac{2080512}{1265}X^2_{a_1a_2a_3a_4}+\frac{1400320}{253}Y^2_{a_1a_2a_3a_4}+\\
&+\frac{6266880}{253}Y^{2\prime}{}_{a_1a_2a_3a_4}-\frac{488960}{3289}A^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}+\\
&+\frac{2228480}{9867}A'^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}+\frac{14}{11}D\Gamma_{a_1a_2a_3a_4}\tilde{S}+\\
&+896A'_{a_1a_2a_3a_4}\,,
\end{aligned}
\tag{B.56}
$$

$$
\begin{aligned}
3(\Gamma_e)^{\alpha\beta}(\Gamma^e{}_{a_1a_2a_3a_4})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& -42(\Gamma_{[a_1a_2|})^{\alpha\beta}R_{\alpha\beta|a_3a_4]}+\frac{3}{2}(\Gamma_{a_1a_2a_3a_4}{}^{bc})^{\alpha\beta}R_{\alpha\beta bc}\\
=&\frac{2080512}{253}X^2_{a_1a_2a_3a_4}+\frac{7001600}{253}Y^2_{a_1a_2a_3a_4}+\\
&+\frac{31334400}{253}Y^{2\prime}{}_{a_1a_2a_3a_4}+\frac{3449088}{3289}A^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}+\\
&+\frac{907520}{3289}A'^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}+\frac{70}{11}D\Gamma_{a_1a_2a_3a_4}\tilde{S}+\\
&+3584A_{a_1a_2a_3a_4}+9856A'_{a_1a_2a_3a_4}\,. \qquad\qquad (B.57)
\end{aligned}
$$

From eqs. (B.53)–(B.57) we get

$$
\begin{aligned}
A_{a_1a_2a_3a_4} =& -2A'_{a_1a_2a_3a_4}-\frac{1}{1408}D\Gamma_{a_1a_2a_3a_4}\tilde{S}-\\
&-\frac{1161}{1265}X^2_{a_1a_2a_3a_4}-\frac{5470}{1771}Y^2_{a_1a_2a_3a_4}-\\
&-\frac{24480}{1771}Y^{2\prime}{}_{a_1a_2a_3a_4}-\frac{21783}{46046}A^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}\,,\\
A^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]} =& -2A'^{i_1i_2i_3}{}_{[a_1|}Y_{i_1i_2i_3|a_2a_3,a_4]}\,. \qquad\qquad (B.58)
\end{aligned}
$$

Note that the second equation is implied by the first one.

The 5-forms. From the 1st SSBI we get

$$
\begin{aligned}
(\Gamma_{[a_1a_2a_3a_4|}{}^b)^{\alpha\beta}R_{\alpha\beta b|a_5]} =& 2D\Gamma_{a_1\dots a_5}\tilde{S}-\frac{69888}{115}X^2_{a_1\dots a_5}+\frac{49920}{161}Y^2_{a_1\dots a_5}-\\
&-\frac{99840}{161}Y^{2\prime}_{a_1\dots a_5}+\frac{1536}{115}D^eY_{a_1\dots a_5,e}+\\
&+128A_{a_1\dots a_5}-\frac{2272}{1035}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]}+\\
&+256A_{[a_1a_2a_3}{}^iX_{a_4a_5],i}+\\
&+768A'_{a_1\dots a_5}+\frac{2656}{1035}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]}-\\
&-\frac{44288}{23}A'_{[a_1a_2a_3}{}^iX_{a_4a_5],i}\,, \qquad\qquad (B.59)\\
(\Gamma_{[a_1a_2a_3a_4|}{}^c)^{\alpha\beta}R_{\alpha\beta|a_5]c} =& 2D\Gamma_{a_1\dots a_5}\tilde{S}-\frac{69888}{115}X^2_{a_1\dots a_5}+\frac{49920}{161}Y^2_{a_1\dots a_5}-\\
&-\frac{99840}{161}Y^{2\prime}_{a_1\dots a_5}+\frac{64}{115}D^eY_{a_1\dots a_5,e}+\\
&+896A_{a_1\dots a_5}+\frac{224}{345}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]}-\\
&-768A_{[a_1a_2a_3}{}^iX_{a_4a_5],i}-\\
&-\frac{832}{345}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]}-\frac{3072}{23}A'_{[a_1a_2a_3}{}^iX_{a_4a_5],i}\,,\\
&\qquad\qquad (B.60)
\end{aligned}
$$

$$
\begin{aligned}
0=\delta^{bc}(\Gamma_{a_1\dots a_5})^{\alpha\beta}R_{\alpha\beta bc} =& 22D\Gamma_{a_1\dots a_5}\tilde{S}+64D^eY_{a_1\dots a_5,e}+\\
&+640A_{a_1\dots a_5}-\frac{32}{9}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]}+
\end{aligned}
$$

$$+ 1280A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i} -$$
$$- 768A'_{a_1\dots a_5} + \frac{32}{9}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]} -$$
$$- 1280A'_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}\,. \tag{B.61}$$

From the 2nd SSBI we get

$$\begin{aligned}
3(\Gamma_{[a_1})^{\alpha\beta}(\Gamma_{a_2a_3a_4a_5]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& -3(\Gamma_{[a_1a_2a_3a_4|}{}^{c})^{\alpha\beta}R_{\alpha\beta|a_5]c}\\
=&\frac{10}{11}D\Gamma_{a_1\dots a_5}\tilde{S} + \frac{466176}{1265}X^2_{a_1\dots a_5} - \frac{9795840}{1771}Y^2_{a_1\dots a_5} -\\
&- \frac{9553920}{1771}Y^{2\prime}_{a_1\dots a_5} - \frac{179456}{16445}D^eY_{a_1\dots a_5,e} +\\
&+ 1664A_{a_1\dots a_5} + \frac{430432}{148005}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]} +\\
&+ 256A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i} -\\
&- 768A'_{a_1\dots a_5} + \frac{135584}{148005}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]} +\\
&+ \frac{6217472}{3289}A'_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}\,,
\end{aligned} \tag{B.62}$$

$$\begin{aligned}
3(\Gamma_{[a_1a_2})^{\alpha\beta}(\Gamma_{a_3a_4a_5]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& -(\Gamma_{[a_1a_2a_3a_4|}{}^{c})^{\alpha\beta}R_{\alpha\beta|a_5]c}\\
=&2D\Gamma_{a_1\dots a_5}\tilde{S} - \frac{69888}{115}X^2_{a_1\dots a_5} + \frac{49920}{161}Y^2_{a_1\dots a_5} -\\
&- \frac{99840}{161}Y^{2\prime}_{a_1\dots a_5} + \frac{1536}{115}D^eY_{a_1\dots a_5,e} +\\
&+ 128A_{a_1\dots a_5} - \frac{2272}{1035}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]} +\\
&+ 1792A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i} +\\
&+ 768A'_{a_1\dots a_5} + \frac{2656}{1035}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]} +\\
&+ \frac{26368}{23}A'_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}\,,
\end{aligned} \tag{B.63}$$

$$\begin{aligned}
3(\Gamma^e)^{\alpha\beta}(\Gamma_{ea_1a_2a_3a_4a_5})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& -20(\Gamma_{[a_1a_2a_3a_4|}{}^{c})^{\alpha\beta}R_{\alpha\beta|a_5]c}\\
=&\frac{60}{11}D\Gamma_{a_1\dots a_5}\tilde{S} + \frac{2380800}{253}X^2_{a_1\dots a_5} - \frac{65280000}{1771}Y^2_{a_1\dots a_5} -\\
&- \frac{44313600}{1771}Y^{2\prime}_{a_1\dots a_5} - \frac{217600}{3289}D^eY_{a_1\dots a_5,e} +\\
&+ 3840A_{a_1\dots a_5} + \frac{99520}{9867}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]} -\\
&- 7680A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i} -\\
&- 9216A'_{a_1\dots a_5} - \frac{20736}{3289}\epsilon_{[a_1a_2a_3|i_1\dots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\dots i_8}{}_{,|a_5]} -\\
&- \frac{29383680}{3289}A'_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}\,,
\end{aligned} \tag{B.64}$$

$$\begin{aligned}
3(\Gamma_{a_1a_2a_3a_4a_5})^{\alpha\beta}\delta_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta =& 5(\Gamma_{[a_1a_2a_3a_4|}{}^{c})^{\alpha\beta}R_{\alpha\beta|a_5]c}\\
=&\frac{386}{11}D\Gamma_{a_1\dots a_5}\tilde{S} - \frac{258816}{253}X^2_{a_1\dots a_5} + \frac{10387200}{1771}Y^2_{a_1\dots a_5} +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{8371200}{1771}Y^{2\prime}_{a_1\ldots a_5}+\frac{342272}{3289}D^eY_{a_1\ldots a_5,e}+\\
&+640A_{a_1\ldots a_5}-\frac{123680}{29601}\epsilon_{[a_1a_2a_3|i_1\ldots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\ldots i_8}{}_{,|a_5]}+\\
&+1280A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}-\\
&-768A'_{a_1\ldots a_5}+\frac{119456}{29601}\epsilon_{[a_1a_2a_3|i_1\ldots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\ldots i_8}{}_{,|a_5]}-\\
&-\frac{4747520}{3289}A'_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}\,. \qquad \text{(B.65)}
\end{aligned}
$$

The following identities are useful

$$
\begin{aligned}
\epsilon_{i_1\ldots i_8[a_1a_2a_3}A^{i_1i_2i_3}{}_{a_4}Y^{i_4\ldots i_8}{}_{,a_5]} &= -\frac{5}{4}\epsilon_{i_1\ldots i_7[a_1\ldots a_4|}A^{i_1i_2i_3}{}_{j}Y^{ji_4\ldots i_7}{}_{,|a_5]}\\
&= \frac{5}{4}\epsilon_{i_1\ldots i_8[a_1a_2a_3}A^{i_1\ldots i_4}Y^{i_5\ldots i_8}{}_{a_4,a_5]}\\
&= \frac{15}{4}\epsilon_{i_1\ldots i_7[a_1\ldots a_4}A_{a_5]}{}^{i_1i_2j}Y_j{}^{i_3\ldots i_6,i_7}\\
&= -\frac{3}{2}\epsilon_{i_1\ldots i_6a_1\ldots a_5}A^{i_1i_2jk}Y_{jk}{}^{i_3i_4i_5,i_6}\\
&= \frac{5}{2}\epsilon_{i_1\ldots i_7[a_1\ldots a_4|}A^{i_1i_2i_3j}Y_{j|a_5]}{}^{i_4i_5i_6,i_7}\,. \qquad \text{(B.66)}
\end{aligned}
$$

From eqs. (B.59)–(B.65) we get

$$
\begin{aligned}
A_{a_1a_2a_3a_4a_5}=&\frac{11}{160264}(10272A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}-29\epsilon_{[a_1a_2a_3|i_1\ldots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\ldots i_8}{}_{,|a_5]})+\\
&+\frac{120}{20033}D^eY_{a_1\ldots a_5,e}+\frac{6}{53935}(4151X^2_{a_1\ldots a_5}+8150Y^2_{a_1\ldots a_5}+15325Y^{2\prime}_{a_1\ldots a_5})\,,\\
A'_{a_1a_2a_3a_4a_5}=&-\frac{1}{2884752}(2569320A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}-7819\epsilon_{[a_1a_2a_3|i_1\ldots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\ldots i_8}{}_{,|a_5]})-\\
&-\frac{681}{80132}D^eY_{a_1\ldots a_5,e}+\frac{2}{10787}(4732X^2_{a_1\ldots a_5}-25(343Y^2_{a_1\ldots a_5}+73Y^{2\prime}_{a_1\ldots a_5}))\,,\\
\frac{11}{8}D\Gamma_{a_1\ldots a_5}\tilde{S}=&-4D^eY_{a_1\ldots a_5,e}-120A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}+\frac{1}{3}\epsilon_{[a_1a_2a_3|i_1\ldots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\ldots i_8}{}_{,|a_5]}-\\
&-40A_{a_1a_2a_3a_4a_5}+48A'_{a_1a_2a_3a_4a_5} \qquad \text{(B.67)}
\end{aligned}
$$

and

$$
\begin{aligned}
\epsilon_{[a_1a_2a_3|i_1\ldots i_8}A^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\ldots i_8}{}_{,|a_5]} &= -2\epsilon_{[a_1a_2a_3|i_1\ldots i_8}A'^{i_1i_2i_3}{}_{|a_4|}Y^{i_4\ldots i_8}{}_{,|a_5]}\,,\\
A_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i} &= -2A'_{[a_1a_2a_3}{}^{i}X_{a_4a_5],i}\,. \qquad \text{(B.68)}
\end{aligned}
$$

Note that the last two equations are redundant, as they are implied by the zeroth order relation

$$
A_{i_1\ldots i_4}=-2A'_{i_1\ldots i_4}\,. \qquad \text{(B.69)}
$$

The (20000)**'s.** From the 1st SSBI we get

$$
\Pi\big[(\Gamma_a{}^c)^{\alpha\beta}R_{\alpha\beta bc}\big]=\frac{33792}{23}X^2_{a,b}+\frac{56320}{23}Y^2_{a,b}-1280B_{a,b}+
$$

$$+\frac{992}{207}A'^{i_1\dots i_4}Y_{i_1\dots i_4(a,b)}+\frac{4544}{207}D^eX_{e(a,b)}\,, \tag{B.70}$$

$$\begin{aligned}\Pi\big[(\Gamma_a{}^c)^{\alpha\beta}R_{\alpha\beta cb}\big] = &\frac{33792}{23}X^2_{a,b}+\frac{56320}{23}Y^2_{a,b}-128B_{a,b}-\\ &-\frac{64}{3}A^{i_1\dots i_4}Y_{i_1\dots i_4(a,b)}+\frac{6512}{207}A'^{i_1\dots i_4}Y_{i_1\dots i_4(a,b)}+\\ &+\frac{17792}{207}D^eX_{e(a,b)}\,,\end{aligned} \tag{B.71}$$

where we have denoted by Π the projection onto the hook-irreducible part. See appendix C for a detailed discussion.

From the 2nd SSBI we get

$$\begin{aligned}3\Pi\big[(\Gamma_a)^{\alpha\beta}(\Gamma_b)_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] =& -2\Pi\big[(\Gamma_a{}^c)^{\alpha\beta}R_{\alpha\beta bc}\big]\\ =& -\frac{55296}{23}X^2_{a,b}-\frac{92160}{23}Y^2_{a,b}+2304B_{a,b}+\\ &+\frac{3616}{759}A'^{i_1\dots i_4}Y_{i_1\dots i_4(a,b)}-\frac{6144}{253}D^eX_{e(a,b)}\,.\end{aligned} \tag{B.72}$$

Implementing the zeroth-order relation $A_{a_1\dots a_4}=-2A'_{a_1\dots a_4}$ we get

$$B_{a,b}=-\frac{1021}{36432}A^{i_1\dots i_4}Y_{i_1\dots i_4(a,b)}+\frac{349}{4554}D^eX_{e(a,b)}+\frac{48}{23}X^2_{a,b}+\frac{80}{23}Y^2_{a,b}\,. \tag{B.73}$$

The (11000)**'s.** From the 1st SSBI we get

$$\Pi\big[(\Gamma_b)^{\alpha\beta}R_{\alpha\beta a_1a_2}\big]=64B_{a_1a_2,b}+\frac{64}{3}A\circ X_{a_1a_2,b}+\frac{2}{135}A'\circ Y_{a_1a_2,b}\,, \tag{B.74}$$

$$\begin{aligned}\Pi\big[(\Gamma_{[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2]b}\big] =& -\frac{1}{2}\Pi\big[(\Gamma_b)^{\alpha\beta}R_{\alpha\beta a_1a_2}\big]\\ =&\,64B_{a_1a_2,b}-\frac{6400}{23}X^2_{a_1a_2,b}-\frac{256}{69}X^{2\prime}_{a_1a_2,b}-\\ &-\frac{14080}{69}Y^2_{a_1a_2,b}-\frac{89600}{23}Y^{2\prime}_{a_1a_2,b}-\\ &-\frac{3008}{69}A\circ X_{a_1a_2,b}-\frac{2}{69}A'\circ Y_{a_1a_2,b}\,,\end{aligned} \tag{B.75}$$

where

$$\begin{aligned}A\circ X_{a_1a_2,b}&:=A_{a_1a_2}{}^{i_1i_2}X_{i_1i_2,b}-A_{b[a_1|}{}^{i_1i_2}X_{i_1i_2,|a_2]}\\ A'\circ Y_{a_1a_2,b}&:=\epsilon_{a_1a_2}{}^{i_1\dots i_9}A'_{i_1\dots i_4}Y_{i_5\dots i_9,b}-\epsilon_{b[a_1|}{}^{i_1\dots i_9}A'_{i_1\dots i_4}Y_{i_5\dots i_9,|a_2]}\,.\end{aligned} \tag{B.76}$$

From the 2nd SSBI we get

$$\begin{aligned}3\Pi\big[(\Gamma_b)^{\alpha\beta}(\Gamma_{a_1a_2})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] =& -18\Pi\big[(\Gamma_b)^{\alpha\beta}R_{\alpha\beta a_1a_2}\big]\\ =& -2304B_{a_1a_2,b}+\frac{76800}{23}X^2_{a_1a_2,b}+\frac{1024}{23}X^{2\prime}_{a_1a_2,b}+\\ &+\frac{56320}{23}Y^2_{a_1a_2,b}+\frac{1075200}{23}Y^{2\prime}_{a_1a_2,b}+\\ &+\frac{256}{23}A\circ X_{a_1a_2,b}-\frac{8}{1035}A'\circ Y_{a_1a_2,b}\,,\end{aligned} \tag{B.77}$$

$$\begin{aligned}3\Pi\big[(\Gamma_{b[a_1})^{\alpha\beta}(\Gamma_{a_2]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] &= \Pi\big[(\Gamma_b)^{\alpha\beta}R_{\alpha\beta a_1a_2}\big]\\ &= 128B_{a_1a_2,b} - \frac{217600}{3289}X^2_{a_1a_2,b} - \frac{269824}{3289}X^{2\prime}_{a_1a_2,b}+\\ &\quad + \frac{468480}{3289}Y^2_{a_1a_2,b} - \frac{19532800}{3289}Y^{2\prime}_{a_1a_2,b}-\\ &\quad - \frac{172928}{9867}A\circ X_{a_1a_2,b} + \frac{508}{40365}A'\circ Y_{a_1a_2,b}\,. \qquad \text{(B.78)}\end{aligned}$$

We get

$$\begin{aligned}B_{a_1a_2,b} &= \frac{1327}{2815200}A'\circ Y_{a_1a_2,b} + \frac{2080}{391}X^2_{a_1a_2,b} - \frac{616}{391}X^{2\prime}_{a_1a_2,b}+\\ &\quad + \frac{3040}{391}Y^2_{a_1a_2,b} - \frac{10640}{391}Y^{2\prime}_{a_1a_2,b}\\ A\circ X_{a_1a_2,b} &= \frac{11}{15300}A'\circ Y_{a_1a_2,b} + \frac{120}{17}X^2_{a_1a_2,b} - \frac{80}{17}X^{2\prime}_{a_1a_2,b}+\\ &\quad + \frac{280}{17}Y^2_{a_1a_2,b} - \frac{3360}{17}Y^{2\prime}_{a_1a_2,b}\,, \qquad \text{(B.79)}\end{aligned}$$

The (10100)**'s.** From the 1st SSBI we get

$$\begin{aligned}\Pi\big[(\Gamma_{b[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2a_3]}\big] &= \frac{128}{3}B_{a_1a_2a_3,b} - 32\left(D_{[a_1}X_{a_2a_3],b} - \frac{1}{9}\eta_{b[a_1}D^iX_{a_2a_3],i}\right)+\\ &\quad + \frac{64}{3}A\circ Y^{(1)}_{a_1a_2a_3,b} - \frac{16}{3}A\circ Y^{(2)}_{a_1a_2a_3,b} + \frac{32}{3}A'\circ Y^{(1)}_{a_1a_2a_3,b}\,, \qquad \text{(B.80)}\\ \Pi\big[(\Gamma_{[a_1a_2|})^{\alpha\beta}R_{\alpha\beta|a_3]b}\big] &= \Pi\big[(\Gamma_{b[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2a_3]}\big]\\ &= -\frac{128}{3}B_{a_1a_2a_3,b} - \frac{4864}{115}X^2_{a_1a_2a_3,b} - \frac{8192}{345}X^{2\prime}_{a_1a_2a_3,b}\\ &\quad - \frac{2048}{23}Y^2_{a_1a_2a_3,b} - \frac{14848}{69}Y^{2\prime}_{a_1a_2a_3,b} - \frac{35840}{23}Y^{2\prime\prime}_{a_1a_2a_3,b}-\\ &\quad - \frac{1504}{23}\left(D_{[a_1}X_{a_2a_3],b} - \frac{1}{9}\eta_{b[a_1}D^iX_{a_2a_3],i}\right)+\\ &\quad + \frac{32}{3}A\circ Y^{(1)}_{a_1a_2a_3,b} - \frac{2608}{621}A\circ Y^{(2)}_{a_1a_2a_3,b}-\\ &\quad - \frac{1504}{69}A'\circ Y^{(1)}_{a_1a_2a_3,b} + \frac{5696}{621}A'\circ Y^{(2)}_{a_1a_2a_3,b}\,, \qquad \text{(B.81)}\end{aligned}$$

where we have defined

$$\begin{aligned}A\circ Y^{(1)}_{a_1a_2a_3,b} &:= A_{[a_1|}{}^{i_1i_2i_3}Y_{i_1i_2i_3|a_2a_3],b} + A_b{}^{i_1i_2i_3}Y_{i_1i_2i_3[a_1a_2,a_3]} + \frac{1}{9}\eta_{b[a_1|}A^{i_1\ldots i_4}Y_{i_1\ldots i_4|a_2,a_3]}\,,\\ A\circ Y^{(2)}_{a_1a_2a_3,b} &:= 5A_{[a_1|}{}^{i_1i_2i_3}Y_{i_1i_2i_3|a_2a_3],b} - 2A_{[a_1|}{}^{i_1i_2i_3}Y_{i_1i_2i_3b|a_2,a_3]} + 3A_b{}^{i_1i_2i_3}Y_{i_1i_2i_3[a_1a_2,a_3]}\,.\end{aligned}$$
(B.82)

Note that indeed there are two (10100)'s in the decomposition of the tensor product $A_{a_1\ldots a_4}\otimes Y_{b_1\ldots b_5,c}\sim(00010)\otimes(10002)$.

From the 2nd SSBI we get

$$3\Pi\big[(\Gamma_b)^{\alpha\beta}(\Gamma_{a_1a_2a_3})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] = -6\Pi\big[(\Gamma_{b[a_1|})^{\alpha\beta}R_{\alpha\beta|a_2a_3]}\big]$$

$$
\begin{aligned}
= & -2304 B_{a_1a_2a_3,b} + \frac{8036352}{16445} X^2_{a_1a_2a_3,b} + \frac{15357952}{16445} X^{2\prime}_{a_1a_2a_3,b} - \\
& - \frac{2082816}{3289} Y^2_{a_1a_2a_3,b} - \frac{14513152}{3289} Y^{2\prime}_{a_1a_2a_3,b} - \frac{43868160}{3289} Y^{2\prime\prime}_{a_1a_2a_3,b} - \\
& - \frac{149568}{3289} \left(D_{[a_1} X_{a_2a_3],b} - \frac{1}{9} \eta_{b[a_1} D^i X_{a_2a_3],i} \right) + \\
& + 192 A \circ Y^{(1)}_{a_1a_2a_3,b} - \frac{404000}{9867} A \circ Y^{(2)}_{a_1a_2a_3,b} + \\
& + \frac{113472}{253} A' \circ Y^{(1)}_{a_1a_2a_3,b} - \frac{273664}{3289} A' \circ Y^{(2)}_{a_1a_2a_3,b} ,
\end{aligned} \tag{B.83}
$$

$$
\begin{aligned}
3\Pi\big[(\Gamma_{b[a_1})^{\alpha\beta} (\Gamma_{a_2a_3]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] & = -16\Pi\big[(\Gamma_{b[a_1|})^{\alpha\beta} R_{\alpha\beta|a_2a_3]}\big] \\
= & - \frac{512}{3} B_{a_1a_2a_3,b} - \frac{20556288}{16445} X^2_{a_1a_2a_3,b} - \frac{22386688}{16445} X^{2\prime}_{a_1a_2a_3,b} - \\
& - \frac{3188736}{3289} Y^2_{a_1a_2a_3,b} + \frac{1773568}{3289} Y^{2\prime}_{a_1a_2a_3,b} - \frac{48384000}{3289} Y^{2\prime\prime}_{a_1a_2a_3,b} + \\
& + \frac{488192}{3289} (D_{[a_1} X_{a_2a_3],b} - \frac{1}{9} \eta_{b[a_1} D^i X_{a_2a_3],i}) - \\
& - 128 A \circ Y^{(1)}_{a_1a_2a_3,b} + \frac{289600}{9867} A \circ Y^{(2)}_{a_1a_2a_3,b} + \\
& + \frac{46336}{253} A' \circ Y^{(1)}_{a_1a_2a_3,b} - \frac{165632}{3289} A' \circ Y^{(2)}_{a_1a_2a_3,b} .
\end{aligned} \tag{B.84}
$$

Implementing the zeroth-order relation $A_{a_1\dots a_4} = -2A'_{a_1\dots a_4}$ we get

$$
\begin{aligned}
B_{a_1a_2a_3,b} = & \frac{495}{47656} A \circ Y^{(1)}_{a_1a_2a_3,b} + \frac{351}{6808} DX_{a_1a_2a_3,b} + \frac{2916}{4255} X^2_{a_1a_2a_3,b} + \frac{167}{185} X^{2\prime}_{a_1a_2a_3,b} + \\
& + \frac{117}{851} Y^2_{a_1a_2a_3,b} - \frac{3287}{1702} Y^{2\prime}_{a_1a_2a_3,b} + \frac{540}{851} Y^{2\prime\prime}_{a_1a_2a_3,b} , \\
A \circ Y^{(2)}_{a_1a_2a_3,b} = & \frac{351}{259} A \circ Y^{(1)}_{a_1a_2a_3,b} - \frac{405}{37} DX_{a_1a_2a_3,b} - \frac{1080}{37} X^2_{a_1a_2a_3,b} - \frac{1080}{37} X^{2\prime}_{a_1a_2a_3,b} - \\
& - \frac{1080}{37} Y^2_{a_1a_2a_3,b} - \frac{540}{37} Y^{2\prime}_{a_1a_2a_3,b} - \frac{17280}{37} Y^{2\prime\prime}_{a_1a_2a_3,b} ,
\end{aligned} \tag{B.85}
$$

where

$$
DX_{a_1a_2a_3,b} := D_{[a_1} X_{a_2a_3],b} - \frac{1}{9} \eta_{b[a_1} D^i X_{a_2a_3],i} . \tag{B.86}
$$

The (10010)**'s.** From the 1st SSBI we get

$$
\begin{aligned}
\Pi\big[(\Gamma_{a_1\dots a_4}{}^c)^{\alpha\beta} R_{\alpha\beta bc}\big] = & 896 B_{a_1\dots a_4,b} - \frac{43008}{115} X^2_{a_1\dots a_4,b} - 512 X^{2\prime}_{a_1\dots a_4,b} + \\
& + \frac{87040}{161} Y^2_{a_1\dots a_4,b} + \frac{122880}{161} Y^{2\prime}_{a_1\dots a_4,b} + \frac{378880}{483} Y^{2\prime\prime}_{a_1\dots a_4,b} + \\
& + \frac{51200}{483} Y^{2\prime\prime\prime}_{a_1\dots a_4,b} + \frac{25088}{69} A \circ X^{(1)}_{a_1\dots a_4,b} - \frac{19456}{115} A \circ X^{(2)}_{a_1\dots a_4,b} + \\
& + \frac{39680}{69} A' \circ X^{(1)}_{a_1\dots a_4,b} - \frac{13568}{115} A' \circ X^{(2)}_{a_1\dots a_4,b} - \\
& - \frac{1984}{1035} A \circ Y^{(1)}_{a_1\dots a_4,b} + \frac{1312}{3105} A \circ Y^{(2)}_{a_1\dots a_4,b} - \\
& - \frac{560}{207} A' \circ Y^{(1)}_{a_1\dots a_4,b} + \frac{64}{115} A' \circ Y^{(2)}_{a_1\dots a_4,b} +
\end{aligned}
$$

$$+\frac{9344}{1035}\left(D^iY_{ia_1\ldots a_4,b}-D^iY_{ib[a_1\ldots,a_4]}\right), \tag{B.87}$$

$$\begin{aligned}
\Pi\big[(\Gamma_{a_1\ldots a_4}{}^c)^{\alpha\beta}R_{\alpha\beta cb}\big] =\, & 128B_{a_1\ldots a_4,b}-\frac{43008}{115}X^2_{a_1\ldots a_4,b}-512X^{2\prime}_{a_1\ldots a_4,b}+\\
&+\frac{87040}{161}Y^2_{a_1\ldots a_4,b}+\frac{122880}{161}Y^{2\prime}_{a_1\ldots a_4,b}+\frac{378880}{483}Y^{2\prime\prime}_{a_1\ldots a_4,b}+\\
&+\frac{51200}{483}Y^{2\prime\prime\prime}_{a_1\ldots a_4,b}-\frac{10240}{69}A\circ X^{(1)}_{a_1\ldots a_4,b}+\frac{15872}{115}A\circ X^{(2)}_{a_1\ldots a_4,b}+\\
&+\frac{4352}{69}A'\circ X^{(1)}_{a_1\ldots a_4,b}+\frac{233728}{115}A'\circ X^{(2)}_{a_1\ldots a_4,b}+\\
&+\frac{224}{1035}A\circ Y^{(1)}_{a_1\ldots a_4,b}+\frac{5728}{3105}A\circ Y^{(2)}_{a_1\ldots a_4,b}-\\
&-\frac{592}{1035}A'\circ Y^{(1)}_{a_1\ldots a_4,b}-\frac{2368}{1035}A'\circ Y^{(2)}_{a_1\ldots a_4,b}+\\
&+\frac{62336}{1035}\left(D^iY_{ia_1\ldots a_4,b}-D^iY_{ib[a_1\ldots,a_4]}\right),
\end{aligned} \tag{B.88}$$

where we have defined

$$\begin{aligned}
A\circ X^{(1)}_{a_1\ldots a_4,b} &= A_{[a_1a_2a_3|}{}^iX_{i|a_4],b}+A_{[a_1a_2a_3|}{}^iX_{ib,|a_4]}+\frac{9}{16}\eta_{b[a_1}A_{a_2a_3|}{}^{ij}X_{ij,|a_4]}\\
A\circ X^{(2)}_{a_1\ldots a_4,b} &= \frac{4}{3}A_{[a_1a_2a_3|}{}^iX_{i|a_4],b}+\frac{1}{3}A_{[a_1a_2a_3|}{}^iX_{ib,|a_4]}+\frac{1}{2}A_{b[a_1a_2}{}^iX_{a_3a_4],i}+\\
&\quad+\frac{5}{16}\eta_{b[a_1}A_{a_2a_3|}{}^{ij}X_{ij,|a_4]}
\end{aligned} \tag{B.89}$$

and

$$\begin{aligned}
A\circ Y^{(1)}_{a_1\ldots a_4,b} :=\, & \Pi(\epsilon_{b[a_1a_2|}{}^{i_1\ldots i_8}A_{i_1i_2i_3|a_3|}Y_{i_4\ldots i_8,|a_4]})\\
=\, & -\frac{2}{5}\epsilon_{b[a_1a_2}{}^{i_1\ldots i_8}A_{a_3|i_1i_2i_3}Y_{i_4\ldots i_8,|a_4]}+\frac{1}{5}\epsilon_{[a_1a_2a_3}{}^{i_1\ldots i_8}A_{a_4]i_1i_2i_3}Y_{i_4\ldots i_8,b}-\\
& -\frac{1}{5}\epsilon_{[a_1a_2a_3|}{}^{i_1\ldots i_8}A_{bi_1i_2i_3}Y_{i_4\ldots i_8,|a_4]}+\frac{1}{8}\eta_{b[a_1}\epsilon_{a_2a_3|}{}^{i_1\ldots i_9}A_{i_1\ldots i_4}Y_{i_5\ldots i_9,|a_4]}\\
A\circ Y^{(2)}_{a_1\ldots a_4,b} :=\, & \Pi(\epsilon_{[a_1a_2a_3}{}^{i_1\ldots i_8}A_{a_4]i_1i_2i_3}Y_{i_4\ldots i_8,b})\\
=\, & -\frac{3}{5}\epsilon_{b[a_1a_2}{}^{i_1\ldots i_8}A_{a_3|i_1i_2i_3}Y_{i_4\ldots i_8,|a_4]}+\frac{4}{5}\epsilon_{[a_1a_2a_3}{}^{i_1\ldots i_8}A_{a_4]i_1i_2i_3}Y_{i_4\ldots i_8,b}+\\
& +\frac{1}{5}\epsilon_{[a_1a_2a_3|}{}^{i_1\ldots i_8}A_{bi_1i_2i_3}Y_{i_4\ldots i_8,|a_4]}+\frac{3}{32}\eta_{b[a_1}\epsilon_{a_2a_3|}{}^{i_1\ldots i_9}A_{i_1\ldots i_4}Y_{i_5\ldots i_9,|a_4]}\,.
\end{aligned} \tag{B.90}$$

In deriving (B.87), (B.88) we have used the relations

$$\begin{aligned}
\epsilon_{[a_1a_2|}{}^{i_1\ldots i_9}A_{i_1\ldots i_4}Y_{|a_3]i_5\ldots,i_9} &= \frac{1}{5}\epsilon_{[a_1a_2|}{}^{i_1\ldots i_9}A_{i_1\ldots i_4}Y_{i_5\ldots i_9,|a_3]}\\
&= -\frac{4}{3}\epsilon_{a_1a_2a_3}{}^{i_1\ldots i_8}A_{i_1i_2i_3}{}^jY_{ji_4\ldots,i_8}
\end{aligned} \tag{B.91}$$

and

$$A\circ Y^{(3)} = -\frac{1}{5}A\circ Y^{(1)}+\frac{1}{15}A\circ Y^{(2)}$$

$$
\begin{aligned}
A\circ Y^{(4)} &= -3A\circ Y^{(1)} + A\circ Y^{(2)}\\
A\circ Y^{(5)} &= -\frac{4}{5}A\circ Y^{(1)} + \frac{4}{15}A\circ Y^{(2)}\\
A\circ Y^{(6)} &= \frac{4}{5}A\circ Y^{(1)}\\
A\circ Y^{(7)} &= -\frac{4}{5}A\circ Y^{(2)}\\
A\circ Y^{(8)} &= \frac{12}{5}A\circ Y^{(1)} - \frac{4}{5}A\circ Y^{(2)}\\
A\circ Y^{(9)} &= \frac{3}{20}A\circ Y^{(1)}\\
A\circ Y^{(10)} &= \frac{3}{5}A\circ Y^{(1)}\\
A\circ Y^{(11)} &= \frac{1}{5}A\circ Y^{(2)}\\
A\circ Y^{(12)} &= \frac{4}{5}A\circ Y^{(2)}\,,
\end{aligned}
\tag{B.92}
$$

where

$$
\begin{aligned}
A\circ Y^{(3)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{b[a_1a_2a_3}{}^{i_1\ldots i_7}A^j{}_{a_4]i_1i_2}Y_{ji_3\ldots,i_7})\\
A\circ Y^{(4)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{[a_1a_2a_3|}{}^{i_1\ldots i_8}A_{bi_1i_2i_3}Y_{i_4\ldots i_8,|a_4]})\\
A\circ Y^{(5)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{a_1\ldots a_4}{}^{i_1\ldots i_7}A_{bi_1i_2}{}^jY_{ji_3\ldots,i_7})\\
A\circ Y^{(6)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{b[a_1a_2|}{}^{i_1\ldots i_8}A_{i_1\ldots i_4}Y_{i_5\ldots i_8|a_3,a_4]})\\
A\circ Y^{(7)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{[a_1a_2a_3|}{}^{i_1\ldots i_8}A_{i_1\ldots i_4}Y_{i_5\ldots i_8|a_4],b})\\
A\circ Y^{(8)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{[a_1a_2a_3|}{}^{i_1\ldots i_8}A_{i_1\ldots i_4}Y_{i_5\ldots i_8b,|a_4]})\\
A\circ Y^{(9)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{b[a_1a_2a_3|}{}^{i_1\ldots i_7}A_{i_1i_2i_3}{}^jY_{j|a_4]i_4\ldots,i_7})\\
A\circ Y^{(10)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{a_1\ldots a_4}{}^{i_1\ldots i_7}A^j{}_{i_1i_2i_3}Y_{jbi_4\ldots,i_7})\\
A\circ Y^{(11)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{b[a_1a_2a_3|}{}^{i_1\ldots i_7}A^j{}_{i_1i_2i_3}Y_{ji_4\ldots i_7,|a_4]})\\
A\circ Y^{(12)}_{a_1\ldots a_4,b} &:= \Pi(\epsilon_{a_1\ldots a_4}{}^{i_1\ldots i_7}A_{i_1i_2i_3}{}^jY_{ji_4\ldots i_7,b})\,.
\end{aligned}
\tag{B.93}
$$

Note that there are only two independent $A\circ Y$ structures, as can be seen from the fact that

$$A_{a_1\ldots a_4}\otimes Y_{b_1\ldots b_5,c}\sim(00010)\otimes(10002)=2(10010)\oplus\ldots\,.$$

From the 2nd SSBI we get

$$
\begin{aligned}
3\Pi\big[(\Gamma_b)^{\alpha\beta}(\Gamma_{a_1\ldots a_4})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] &= 2\Pi\big[(\Gamma_{a_1\ldots a_4}{}^c)^{\alpha\beta}R_{\alpha\beta bc}\big]\\
&= 2304B_{a_1\ldots a_4,b} - \frac{786432}{16445}X^2_{a_1\ldots a_4,b} - \frac{66048}{143}X^{2\prime}_{a_1\ldots a_4,b}+\\
&\quad + \frac{24975360}{23023}Y^2_{a_1\ldots a_4,b} + \frac{27985920}{23023}Y^{2\prime}_{a_1\ldots a_4,b}+\\
&\quad + \frac{62423040}{23023}Y^{2\prime\prime}_{a_1\ldots a_4,b} - \frac{25651200}{23023}Y^{2\prime\prime\prime}_{a_1\ldots a_4,b}-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3348480}{3289}A\circ X^{(1)}_{a_1\dots a_4,b}+\frac{19915776}{16445}A\circ X^{(2)}_{a_1\dots a_4,b}-\\
&-\frac{2998784}{3289}A'\circ X^{(1)}_{a_1\dots a_4,b}+\frac{15641088}{16445}A'\circ X^{(2)}_{a_1\dots a_4,b}+\\
&+\frac{39232}{9867}A\circ Y^{(1)}_{a_1\dots a_4,b}-\frac{153152}{148005}A\circ Y^{(2)}_{a_1\dots a_4,b}+\\
&+\frac{139712}{16445}A'\circ Y^{(1)}_{a_1\dots a_4,b}-\frac{24064}{16445}A'\circ Y^{(2)}_{a_1\dots a_4,b}+\\
&+\frac{179456}{16445}(D^iY_{ia_1\dots a_4,b}-D^iY_{ib[a_1\dots,a_4]})\,,
\end{aligned}
\tag{B.94}
$$

$$
\begin{aligned}
3\Pi\big[(\Gamma_{b[a_1})^{\alpha\beta}(\Gamma_{a_2a_3a_4]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big]=&\frac{3}{2}\Pi\big[(\Gamma_{a_1\dots a_4}{}^c)^{\alpha\beta}R_{\alpha\beta bc}\big]\\
=&-192B_{a_1\dots a_4,b}-\frac{26102784}{16445}X^2_{a_1\dots a_4,b}-\frac{197376}{143}X^{2\prime}_{a_1\dots a_4,b}+\\
&+\frac{6059520}{23023}Y^2_{a_1\dots a_4,b}+\frac{23101440}{23023}Y^{2\prime}_{a_1\dots a_4,b}-\\
&-\frac{43576320}{23023}Y^{2\prime\prime}_{a_1\dots a_4,b}+\frac{62284800}{23023}Y^{2\prime\prime\prime}_{a_1\dots a_4,b}+\\
&+\frac{4500480}{3289}A\circ X^{(1)}_{a_1\dots a_4,b}-\frac{14462208}{16445}A\circ X^{(2)}_{a_1\dots a_4,b}-\\
&-\frac{1488768}{3289}A'\circ X^{(1)}_{a_1\dots a_4,b}+\frac{23135616}{16445}A'\circ X^{(2)}_{a_1\dots a_4,b}-\\
&-\frac{44592}{16445}A\circ Y^{(1)}_{a_1\dots a_4,b}+\frac{6832}{16445}A\circ Y^{(2)}_{a_1\dots a_4,b}-\\
&-\frac{6008}{16445}A'\circ Y^{(1)}_{a_1\dots a_4,b}+\frac{11488}{9867}A'\circ Y^{(2)}_{a_1\dots a_4,b}-\\
&-\frac{111808}{16445}(D^iY_{ia_1\dots a_4,b}-D^iY_{ib[a_1\dots,a_4]})\,.
\end{aligned}
\tag{B.95}
$$

Implementing the zeroth-order relation $A_{a_1\dots a_4}=-2A'_{a_1\dots a_4}$ we get

$$
\begin{aligned}
B_{a_1\dots a_4,b}=&\frac{5052}{4255}A\circ X^{(2)}_{a_1\dots a_4,b}+\frac{401}{2042400}A\circ Y^{(1)}_{a_1\dots a_4,b}-\frac{577}{170200}A\circ Y^{(2)}_{a_1\dots a_4,b}-\\
&-\frac{63}{851}DY_{a_1\dots a_4,b}+\frac{3808}{4255}X^2_{a_1\dots a_4,b}+\frac{991}{851}X^{2\prime}_{a_1\dots a_4,b}-\frac{6780}{5957}Y^2_{a_1\dots a_4,b}-\\
&-\frac{9860}{5957}Y^{2\prime}_{a_1\dots a_4,b}-\frac{8800}{5957}Y^{2\prime\prime}_{a_1\dots a_4,b}-\frac{2540}{5957}Y^{2\prime\prime\prime}_{a_1\dots a_4,b}\,,\\
A\circ X^{(1)}_{a_1\dots a_4,b}=&\frac{81}{37}A\circ X^{(2)}_{a_1\dots a_4,b}+\frac{119}{88800}A\circ Y^{(1)}_{a_1\dots a_4,b}-\frac{217}{66600}A\circ Y^{(2)}_{a_1\dots a_4,b}-\frac{7}{111}DY_{a_1\dots a_4,b}+\\
&+\frac{60}{37}X^2_{a_1\dots a_4,b}+\frac{60}{37}X^{2\prime}_{a_1\dots a_4,b}-\frac{30}{37}Y^2_{a_1\dots a_4,b}-\frac{60}{37}Y^{2\prime}_{a_1\dots a_4,b}+\frac{20}{37}Y^{2\prime\prime}_{a_1\dots a_4,b}-\\
&-\frac{80}{37}Y^{2\prime\prime\prime}_{a_1\dots a_4,b}\,,
\end{aligned}
\tag{B.96}
$$

where

$$
DY_{a_1\dots a_4,b}:=D^iY_{ia_1\dots a_4,b}-D^iY_{ib[a_1\dots,a_4]}\,.
\tag{B.97}
$$

The (10002)**'s.** From the 1st SSBI we get

$$
\Pi\big[(\Gamma_{a_1\dots a_5}{}^c)^{\alpha\beta}R_{\alpha\beta bc}\big]=-768B_{a_1\dots a_5,b}-\frac{8960}{23}X^2_{a_1\dots a_5,b}+\frac{4480}{23}X^{2\prime}_{a_1\dots a_5,b}-
$$

$$
\begin{aligned}
&- \frac{13440}{23} Y^{2}_{a_1\dots a_5,b} - \frac{21760}{23} Y^{2\prime}_{a_1\dots a_5,b} - \frac{17920}{23} Y^{2\prime\prime}_{a_1\dots a_5,b} - \\
&- \frac{10240}{23} Y^{2\prime\prime\prime}_{a_1\dots a_5,b} + \frac{8960}{69} Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b} - \\
&- \frac{8320}{69} A\circ Y^{(1)}_{a_1\dots a_5,b} - \frac{53120}{69} A\circ Y^{(2)}_{a_1\dots a_5,b} - \\
&- \frac{20480}{69} A'\circ Y^{(1)}_{a_1\dots a_5,b} - \frac{81920}{69} A'\circ Y^{(2)}_{a_1\dots a_5,b} - \qquad\qquad (B.98)\\
&- \frac{4}{69}(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6} D_{i_1} Y_{i_2\dots i_6,b} + \epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6} D_{i_1} Y_{i_2\dots i_6,|a_5]}) + \\
&+ \frac{440}{207}(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6} A'_{i_1\dots i_4} X_{i_5 i_6,b} + \epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6} A'_{i_1\dots i_4} X_{i_5 i_6,|a_5]})\,,
\end{aligned}
$$

$$
\begin{aligned}
\Pi\big[(\Gamma_{a_1\dots a_5}{}^{c})^{\alpha\beta} R_{\alpha\beta cb}\big] = &- 128 B_{a_1\dots a_5,b} - \frac{8960}{23} X^{2}_{a_1\dots a_5,b} + \frac{4480}{23} X^{2\prime}_{a_1\dots a_5,b} - \\
&- \frac{13440}{23} Y^{2}_{a_1\dots a_5,b} - \frac{21760}{23} Y^{2\prime}_{a_1\dots a_5,b} - \frac{17920}{23} Y^{2\prime\prime}_{a_1\dots a_5,b} - \\
&- \frac{10240}{23} Y^{2\prime\prime\prime}_{a_1\dots a_5,b} + \frac{8960}{69} Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b} + \\
&+ \frac{50560}{69} A\circ Y^{(1)}_{a_1\dots a_5,b} + \frac{35200}{69} A\circ Y^{(2)}_{a_1\dots a_5,b} - \\
&- \frac{145600}{69} A'\circ Y^{(1)}_{a_1\dots a_5,b} - \frac{104000}{69} A'\circ Y^{(2)}_{a_1\dots a_5,b} - \qquad\qquad (B.99)\\
&- \frac{104}{207}(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6} D_{i_1} Y_{i_2\dots i_6,b} + \epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6} D_{i_1} Y_{i_2\dots i_6,|a_5]}) + \\
&+ \frac{80}{9}(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6} A_{i_1\dots i_4} X_{i_5 i_6,b} + \epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6} A_{i_1\dots i_4} X_{i_5 i_6,|a_5]}) - \\
&- \frac{20}{207}(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6} A'_{i_1\dots i_4} X_{i_5 i_6,b} + \epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6} A'_{i_1\dots i_4} X_{i_5 i_6,|a_5]})\,,
\end{aligned}
$$

where we have defined the two irreducible-hook combinations

$$
\begin{aligned}
A\circ Y^{(1)}_{a_1\dots a_5,b} &:= A_{[a_1 a_2|}{}^{i_1 i_2} Y_{i_1 i_2|a_3 a_4 a_5],b} + A_{b[a_1|}{}^{i_1 i_2} Y_{i_1 i_2|a_2\dots,a_5]}\,,\\
A\circ Y^{(2)}_{a_1\dots a_5,b} &:= A_{[a_1 a_2|}{}^{i_1 i_2} Y_{i_1 i_2 b|a_3 a_4,a_5]} - A_{b[a_1|}{}^{i_1 i_2} Y_{i_1 i_2|a_2\dots,a_5]} + \\
&\quad + \frac{4}{7}\eta_{b[a_1} A_{a_2|}{}^{i_1 i_2 i_3} Y_{i_1 i_2 i_3|a_3 a_4,a_5]}\,. \qquad\qquad (B.100)
\end{aligned}
$$

Note that indeed there are two (10002)'s in the decomposition of the tensor product $A_{a_1\dots a_4}\otimes Y_{b_1\dots b_5,c}\sim(00010)\otimes(10002)$.

From the 2nd SSBI we get

$$
\begin{aligned}
3\Pi\big[(\Gamma_b)^{\alpha\beta}(\Gamma_{a_1\dots a_5})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] &= -2\Pi\big[(\Gamma_{a_1\dots a_5}{}^{c})^{\alpha\beta} R_{\alpha\beta bc}\big]\\
&= 2304 B_{a_1\dots a_5,b} + \frac{990720}{3289} X^{2}_{a_1\dots a_5,b} + \\
&\quad + \frac{1388800}{3289} X^{2\prime}_{a_1\dots a_5,b} - \frac{162560}{3289} Y^{2}_{a_1\dots a_5,b} + \\
&\quad + \frac{42984960}{23023} Y^{2\prime}_{a_1\dots a_5,b} + \frac{7398400}{3289} Y^{2\prime\prime}_{a_1\dots a_5,b} + \\
&\quad + \frac{49612800}{23023} Y^{2\prime\prime\prime}_{a_1\dots a_5,b} - \frac{212480}{3289} Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b} +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{51200}{299}A\circ Y^{(1)}_{a_1\dots a_5,b}-\frac{4968960}{3289}A\circ Y^{(2)}_{a_1\dots a_5,b}-\\
&-\frac{197760}{3289}A'\circ Y^{(1)}_{a_1\dots a_5,b}-\frac{836480}{253}A'\circ Y^{(2)}_{a_1\dots a_5,b}+ \qquad\qquad \text{(B.101)}\\
&+\frac{2720}{29601}\left(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6}D_{i_1}Y_{i_2\dots i_6,b}+\epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6}D_{i_1}Y_{i_2\dots i_6,|a_5]}\right)-\\
&-\frac{2920}{897}\left(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6}A'_{i_1\dots i_4}X_{i_5i_6,b}+\epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6}A'_{i_1\dots i_4}X_{i_5i_6,|a_5]}\right),
\end{aligned}
$$

$$
\begin{aligned}
3\Pi\big[(\Gamma_{b[a_1})^{\alpha\beta}(\Gamma_{a_2\dots a_5]})_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] &= -\frac{8}{5}\Pi\big[(\Gamma_{a_1\dots a_5}{}^c)^{\alpha\beta}R_{\alpha\beta bc}\big]\\
&=\frac{1024}{5}B_{a_1\dots a_5,b}+\frac{2936832}{3289}X^2_{a_1\dots a_5,b}-\\
&\quad-\frac{3376128}{3289}X^{2\prime}_{a_1\dots a_5,b}+\frac{5394432}{3289}Y^2_{a_1\dots a_5,b}+\\
&\quad+\frac{18696192}{23023}Y^{2\prime}_{a_1\dots a_5,b}+\frac{1069056}{3289}Y^{2\prime\prime}_{a_1\dots a_5,b}-\\
&\quad-\frac{13953024}{23023}Y^{2\prime\prime\prime}_{a_1\dots a_5,b}-\frac{531456}{3289}Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b}+\\
&\quad+\frac{8192}{23}A\circ Y^{(1)}_{a_1\dots a_5,b}+\frac{2085888}{3289}A\circ Y^{(2)}_{a_1\dots a_5,b}+\\
&\quad+\frac{657920}{3289}A'\circ Y^{(1)}_{a_1\dots a_5,b}-\frac{936448}{3289}A'\circ Y^{(2)}_{a_1\dots a_5,b}- \qquad\qquad \text{(B.102)}\\
&\quad-\frac{8512}{148005}\left(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6}D_{i_1}Y_{i_2\dots i_6,b}+\epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6}D_{i_1}Y_{i_2\dots i_6,|a_5]}\right)-\\
&\quad-\frac{7840}{897}\left(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6}A'_{i_1\dots i_4}X_{i_5i_6,b}+\epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6}A'_{i_1\dots i_4}X_{i_5i_6,|a_5]}\right),
\end{aligned}
$$

$$
\begin{aligned}
3\Pi\big[(\Gamma_{a_1\dots a_5})^{\alpha\beta}(\Gamma_b)_\delta{}^\gamma R_{(\alpha\beta\gamma)}{}^\delta\big] &= -2\Pi\big[(\Gamma_{a_1\dots a_5}{}^c)^{\alpha\beta}R_{\alpha\beta bc}\big]\\
&=256B_{a_1\dots a_5,b}+\frac{2434560}{3289}X^2_{a_1\dots a_5,b}\\
&\quad-\frac{2983680}{3289}X^{2\prime}_{a_1\dots a_5,b}+\frac{8597760}{3289}Y^2_{a_1\dots a_5,b}+\\
&\quad+\frac{40680960}{23023}Y^{2\prime}_{a_1\dots a_5,b}-\frac{3609600}{3289}Y^{2\prime\prime}_{a_1\dots a_5,b}-\\
&\quad-\frac{43407360}{23023}Y^{2\prime\prime\prime}_{a_1\dots a_5,b}-\frac{3755520}{3289}Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b}+\\
&\quad+\frac{192000}{299}A\circ Y^{(1)}_{a_1\dots a_5,b}+\frac{1812480}{3289}A\circ Y^{(2)}_{a_1\dots a_5,b}+\\
&\quad+\frac{734080}{3289}A'\circ Y^{(1)}_{a_1\dots a_5,b}-\frac{5027200}{3289}A'\circ Y^{(2)}_{a_1\dots a_5,b}- \qquad\qquad \text{(B.103)}\\
&\quad-\frac{3232}{29601}\left(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6}D_{i_1}Y_{i_2\dots i_6,b}+\epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6}D_{i_1}Y_{i_2\dots i_6,|a_5]}\right)-\\
&\quad-\frac{13480}{897}\left(\epsilon_{a_1\dots a_5}{}^{i_1\dots i_6}A'_{i_1\dots i_4}X_{i_5i_6,b}+\epsilon_{b[a_1\dots a_4|}{}^{i_1\dots i_6}A'_{i_1\dots i_4}X_{i_5i_6,|a_5]}\right).
\end{aligned}
$$

Implementing the zeroth-order relation $A_{a_1\dots a_4}=-2A'_{a_1\dots a_4}$ we get

$$
\begin{aligned}
B_{a_1\dots a_5,b}= &-\frac{65}{92736}A\circ X_{a_1\dots a_5,b}-\frac{295}{966}A\circ Y^{(1)}_{a_1\dots a_5,b}+\frac{20}{23}X^2_{a_1\dots a_5,b}-\frac{30}{23}X^{2\prime}_{a_1\dots a_5,b}+\\
&+\frac{15}{23}Y^2_{a_1\dots a_5,b}-\frac{165}{161}Y^{2\prime}_{a_1\dots a_5,b}+\frac{15}{23}Y^{2\prime\prime}_{a_1\dots a_5,b}-\frac{55}{161}Y^{2\prime\prime\prime}_{a_1\dots a_5,b}+\frac{5}{6}Y^{2\prime\prime\prime\prime}_{a_1\dots a_5,b},
\end{aligned}
$$

$$
\begin{aligned}
A \circ Y^{(2)}_{a_1 \dots a_5, b} = & -\frac{11}{2016} A \circ X_{a_1 \dots a5, b} - \frac{11}{42} A \circ Y^{(1)}_{a_1 \dots a_5, b} + X^2_{a_1 \dots a_5, b} - X^{2\prime}_{a_1 \dots a_5, b} - \\
& - \frac{5}{2} Y^2_{a_1 \dots a_5, b} - 3 Y^{2\prime}_{a_1 \dots a_5, b} + 5 Y^{2\prime\prime}_{a_1 \dots a_5, b} + 4 Y^{2\prime\prime\prime}_{a_1 \dots a_5, b} + \frac{10}{3} Y^{2\prime\prime\prime\prime}_{a_1 \dots a_5, b} \, , \\
D Y_{a_1 \dots a_5, b} = & \frac{55}{12} A \circ X_{a_1 \dots a5, b} + 3220 A \circ Y^{(1)}_{a_1 \dots a_5, b} - 840 X^2_{a_1 \dots a_5, b} + 840 X^{2\prime}_{a_1 \dots a_5, b} - \\
& - 7980 Y^2_{a_1 \dots a_5, b} - 7560 Y^{2\prime}_{a_1 \dots a_5, b} + 5880 Y^{2\prime\prime}_{a_1 \dots a_5, b} + 6720 Y^{2\prime\prime\prime}_{a_1 \dots a_5, b} + 5600 Y^{2\prime\prime\prime\prime}_{a_1 \dots a_5, b} \, ,
\end{aligned}
\tag{B.104}
$$

where

$$
\begin{aligned}
D Y_{a_1 \dots a_5, b} &:= \epsilon_{a_1 \dots a_5}{}^{i_1 \dots i_6} D_{i_1} Y_{i_2 \dots i_6, b} + \epsilon_{b[a_1 \dots a_4|}{}^{i_1 \dots i_6} D_{i_1} Y_{i_2 \dots i_6, |a_5]} \, , \\
A \circ X_{a_1 \dots a5, b} &:= \epsilon_{a_1 \dots a_5}{}^{i_1 \dots i_6} A_{i_1 \dots i_4} X_{i_5 i_6, b} + \epsilon_{b[a_1 \dots a_4|}{}^{i_1 \dots i_6} A_{i_1 \dots i_4} X_{i_5 i_6, |a_5]} \, .
\end{aligned}
\tag{B.105}
$$

B.3 The dimension-3/2 SSBI's

We consider now the SSBI's at dimension $\frac{3}{2}$, which read

$$
\begin{aligned}
2 R_{\alpha [bc]}{}^d &= D_\alpha T_{bc}{}^d + 2 D_{[b} T_{c]\alpha}{}^d + 2 T_{\alpha [b}{}^E T_{|E|c]}{}^d + T_{bc}{}^E T_{E\alpha}{}^d \\
2 R_{a(\beta\gamma)}{}^\delta &= D_a T_{\beta\gamma}{}^\delta + 2 D_{(\beta} T_{\gamma) a}{}^\delta + 2 T_{a(\beta}{}^E T_{|E|\gamma)}{}^\delta + T_{\beta\gamma}{}^E T_{Ea}{}^\delta
\end{aligned}
\tag{B.106}
$$

In an unconstrained superfield in the representation 4290, there are two spinors at level θ^3. The index structure of the SSBI's at this level also contains two spinor equations. By contracting the first of the SSBI's with $\delta^c_d \Gamma^b$ we find

$$
\begin{aligned}
-(\Gamma^b)_\alpha{}^\beta R_{\beta c b}{}^c = & -\frac{13}{6} (\Gamma^{i_1 i_2 i_3} \tilde{S}^{i_4})_\alpha A_{i_1 \dots i_4} - \frac{13}{6} (\Gamma^{i_1 i_2 i_3} \tilde{S}^{i_4})_\alpha A'_{i_1 \dots i_4} + \\
& + \frac{5}{3} (\Gamma^{i_1 \dots i_4} \tilde{S})_\alpha A_{i_1 \dots i_4} - \frac{35}{12} (\Gamma^{i_1 \dots i_4} \tilde{S})_\alpha A'_{i_1 \dots i_4} + \\
& + 13 D^i \tilde{S}_{i\alpha} - 10 \not{D} \tilde{S}_\alpha - \\
& - 220 \tilde{t}_\alpha + 2 (\Gamma^k \tilde{t}^{i_1 i_2})_\alpha X_{i_1 i_2, k} + \frac{1}{6} (\Gamma^{i_1 \dots i_4} \tilde{t}^{i_5 i_6})_\alpha Y_{i_1 \dots i_5, i_6} \, .
\end{aligned}
\tag{B.107}
$$

Contracting with Γ^{bcd} we get

$$
2 (\Gamma^{bcd})_\alpha{}^\beta R_{\beta bcd} = -1980 \tilde{t}_\alpha - 4 (\Gamma^k \tilde{t}^{i_1 i_2})_\alpha X_{i_1 i_2, k} - \frac{1}{3} (\Gamma^{i_1 \dots i_4} \tilde{t}^{i_5 i_6})_\alpha Y_{i_1 \dots i_5, i_6} \, .
\tag{B.108}
$$

Contracting the second of the SSBI's with $C_{\alpha\delta} (\Gamma^a)^{\beta\gamma}$ and taking the Lorentz condition into account, we get

$$
\begin{aligned}
\frac{1}{2} (\Gamma^{bcd})_\alpha{}^\beta R_{\beta bcd} + & (\Gamma^b)_\alpha{}^\beta R_{\beta c b}{}^c = \\
& = + 22 D_\alpha A - 20 (\Gamma^i D)_\alpha A_i + 2 (\Gamma^{i_1 i_2} D)_\alpha A_{i_1 i_2} + \\
& \quad + (\Gamma^{i_1 i_2 i_3} D)_\alpha A_{i_1 i_2 i_3} - \frac{1}{3} (\Gamma^{i_1 \dots i_4} D)_\alpha A_{i_1 \dots i_4} - \frac{1}{12} (\Gamma^{i_1 \dots i_5} D)_\alpha A_{i_1 \dots i_5} - \\
& \quad - 2 (\Gamma^i D)_\alpha A'_i - 9 (\Gamma^{i_1 i_2} D)_\alpha A'_{i_1 i_2} + \frac{8}{3} (\Gamma^{i_1 i_2 i_3} D)_\alpha A'_{i_1 i_2 i_3} + \\
& \quad + \frac{7}{12} (\Gamma^{i_1 \dots i_4} D)_\alpha A'_{i_1 \dots i_4} - \frac{1}{10} (\Gamma^{i_1 \dots i_5} D)_\alpha A'_{i_1 \dots i_5} -
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{13}{3}(\Gamma^{i_1i_2i_3}\tilde{S}^{i_4})_\alpha A_{i_1\dots i_4}-\frac{13}{3}(\Gamma^{i_1i_2i_3}\tilde{S}^{i_4})_\alpha A'_{i_1\dots i_4}-\\
&-\frac{1}{3}(\Gamma^{i_1\dots i_4}\tilde{S})_\alpha A_{i_1\dots i_4}+\frac{7}{12}(\Gamma^{i_1\dots i_4}\tilde{S})_\alpha A'_{i_1\dots i_4}\,.
\end{aligned}
\tag{B.109}
$$

Contracting with $(\Gamma^{ai})_{\alpha\delta}(\Gamma_i)^{\beta\gamma}$ we get

$$
\begin{aligned}
3(\Gamma^{bcd})_\alpha{}^\beta R_{\beta bcd}-8(\Gamma^b)_\alpha{}^\beta R_{\beta cb}{}^c=&\\
=&-220D_\alpha A+160(\Gamma^iD)_\alpha A_i+16(\Gamma^{i_1i_2}D)_\alpha A_{i_1i_2}+\\
&+6(\Gamma^{i_1i_2i_3}D)_\alpha A_{i_1i_2i_3}-\frac{4}{3}(\Gamma^{i_1\dots i_4}D)_\alpha A_{i_1\dots i_4}-\\
&-\frac{1}{6}(\Gamma^{i_1\dots i_5}D)_\alpha A_{i_1\dots i_5}-20(\Gamma^iD)_\alpha A'_i+54(\Gamma^{i_1i_2}D)_\alpha A'_{i_1i_2}-\\
&-\frac{32}{3}(\Gamma^{i_1i_2i_3}D)_\alpha A'_{i_1i_2i_3}-\frac{7}{6}(\Gamma^{i_1\dots i_4}D)_\alpha A'_{i_1\dots i_4}-\\
&-\frac{52}{3}(\Gamma^{i_1i_2i_3}\tilde{S}^{i_4})_\alpha A_{i_1\dots i_4}+\frac{26}{3}(\Gamma^{i_1i_2i_3}\tilde{S}^{i_4})_\alpha A'_{i_1\dots i_4}-\\
&-\frac{4}{3}(\Gamma^{i_1\dots i_4}\tilde{S})_\alpha A_{i_1\dots i_4}-\frac{7}{6}(\Gamma^{i_1\dots i_4}\tilde{S})_\alpha A'_{i_1\dots i_4}+\\
&+32(\Gamma^{i_1\dots i_4}\tilde{Z}')_\alpha A_{i_1\dots i_4}+64(\Gamma^{i_1i_2i_3}\tilde{Z}'^{i_4})_\alpha A_{i_1\dots i_4}+\\
&+32(\Gamma^{i_1i_2}\tilde{Z}'^{i_3i_4})_\alpha A_{i_1\dots i_4}+112(\Gamma^{i_1\dots i_4}\tilde{Z})_\alpha A'_{i_1\dots i_4}-\\
&-320(\Gamma^{i_1i_2i_3}\tilde{Z}^{i_4})_\alpha A'_{i_1\dots i_4}+320(\Gamma^{i_1i_2}\tilde{Z}^{i_3i_4})_\alpha A'_{i_1\dots i_4}-\\
&-128(\Gamma^{i_1}\tilde{Z}^{i_2i_3i_4})_\alpha A'_{i_1\dots i_4}+16\tilde{Z}_\alpha^{i_1\dots i_4}A'_{i_1\dots i_4}+\\
&+7040\tilde{t}_\alpha\,,
\end{aligned}
\tag{B.110}
$$

where we have used the conventions $\Gamma_{012345678910}=-1$ and $\epsilon_{012345678910}=1$ and also the relation

$$
\Gamma_{a_1\dots a_p}=-(-1)^{(p+1)(p-2)/2}\frac{1}{(11-p)!}\epsilon_{a_1\dots a_p}{}^{a_{p+1}\dots a_{11}}\Gamma_{a_{p+1}\dots a_{11}}\,.
\tag{B.111}
$$

Combining the above equations, eliminating $\tilde{t}_{ij}{}^\alpha$, we finally get

$$
\begin{aligned}
-38060\tilde{t}_\alpha=&-440D_\alpha A+280(\Gamma^iD)_\alpha A_i+68(\Gamma^{i_1i_2}D)_\alpha A_{i_1i_2}+\\
&+28(\Gamma^{i_1i_2i_3}D)_\alpha A_{i_1i_2i_3}-\frac{22}{3}(\Gamma^{i_1\dots i_4}D)_\alpha A_{i_1\dots i_4}-\frac{4}{3}(\Gamma^{i_1\dots i_5}D)_\alpha A_{i_1\dots i_5}-\\
&-80(\Gamma^iD)_\alpha A'_i+72(\Gamma^{i_1i_2}D)_\alpha A'_{i_1i_2}-\frac{16}{3}(\Gamma^{i_1i_2i_3}D)_\alpha A'_{i_1i_2i_3}+\\
&+\frac{7}{3}(\Gamma^{i_1\dots i_4}D)_\alpha A'_{i_1\dots i_4}-(\Gamma^{i_1\dots i_5}D)_\alpha A'_{i_1\dots i_5}-65(\Gamma^{i_1i_2i_3}\tilde{S}^{i_4})_\alpha A_{i_1\dots i_4}+\\
&+13(\Gamma^{i_1i_2i_3}\tilde{S}^{i_4})_\alpha A'_{i_1\dots i_4}-\frac{92}{3}(\Gamma^{i_1\dots i_4}\tilde{S})_\alpha A_{i_1\dots i_4}+\frac{259}{6}(\Gamma^{i_1\dots i_4}\tilde{S})_\alpha A'_{i_1\dots i_4}+\\
&+96(\Gamma^{i_1\dots i_4}\tilde{Z}')_\alpha A_{i_1\dots i_4}+192(\Gamma^{i_1i_2i_3}\tilde{Z}'^{i_4})_\alpha A_{i_1\dots i_4}+96(\Gamma^{i_1i_2}\tilde{Z}'^{i_3i_4})_\alpha A_{i_1\dots i_4}+\\
&+336(\Gamma^{i_1\dots i_4}\tilde{Z})_\alpha A'_{i_1\dots i_4}-960(\Gamma^{i_1i_2i_3}\tilde{Z}^{i_4})_\alpha A'_{i_1\dots i_4}+960(\Gamma^{i_1i_2}\tilde{Z}^{i_3i_4})_\alpha A'_{i_1\dots i_4j}-\\
&-384(\Gamma^{i_1}\tilde{Z}^{i_2i_3i_4})_\alpha A'_{i_1\dots i_4}+48\tilde{Z}_\alpha^{i_1\dots i_4}A'_{i_1\dots i_4}-182D^i\tilde{S}_{i\alpha}+140\not{D}\tilde{S}_\alpha\,.
\end{aligned}
\tag{B.112}
$$

B.4 The dimension-2 SSBI's

We consider now the SSBI's at dimension 2, which read

$$\begin{aligned} R_{[abc]}{}^{d} &= D_{[a}T_{bc]}{}^{d} + T_{[ab}{}^{E}T_{|E|c]}{}^{d}\,, \\ R_{ab\gamma}{}^{\delta} &= 2D_{[a}T_{b]\gamma}{}^{\delta} + D_{\gamma}T_{ab}{}^{\delta} + T_{ab}{}^{E}T_{|E|\gamma}{}^{\delta} + 2T_{\gamma[a}{}^{E}T_{|E|b]}{}^{\delta}\,. \end{aligned} \tag{B.113}$$

We will focus on the representations associated with the Einstein equations, (00000) and (20000), the 4-form equation of motion, (00100), and the 4-form BI, (00002). Only the second SSBI in B.113 contributes to these representations and the first SSBI will therefore not be analysed below.

The (00000) **and** (20000)**.** The second SSBI contains one (00000) and one (20000). They are obtained by contracting with $(\Gamma_{bc})_{\delta}{}^{\gamma}$ and symmetrising in ac,

$$\begin{aligned} 16R_{(a}{}^{b}{}_{c)b} = &-288D_{(a}A_{c)} - 32\eta_{ac}D^{i}A_{i} + 32D^{i}B_{i(a,c)} + 10\eta_{ac}D^{\alpha}\tilde{t}_{\alpha} - 9D\Gamma_{(a}\tilde{t}_{c)} + \\ &+ 2\tilde{S}_{(a,}{}^{i\alpha}\tilde{t}_{c)i\alpha} - 11\tilde{S}_{i}\Gamma_{(a}\tilde{t}_{c)}{}^{i} + 140\tilde{Z}_{(a}{}^{i\alpha}\tilde{t}_{c)i\alpha} + 28\tilde{Z}^{i}\Gamma_{(a}\tilde{t}_{c)i} - 4\tilde{Z}'_{(a}{}^{i\alpha}\tilde{t}_{c)i\alpha} - \\ &- 14\tilde{Z}'^{i}\Gamma_{(a}\tilde{t}_{c)i} - 32A_{(a}{}^{i_1i_2i_3}A_{c)i_1i_2i_3} - 64A'_{(a}{}^{i_1i_2i_3}A'_{c)i_1i_2i_3} + 16\eta_{ac}A'_{i_1\dots i_4}A'^{i_1\dots i_4} + \\ &+ \frac{64}{3}A_{(a|i_1i_2i_3|}B^{i_1i_2i_3}{}_{,c)} + \frac{40}{3}A'_{i_1\dots i_4}B^{i_1\dots i_4}{}_{(a,c)}\,. \end{aligned} \tag{B.114}$$

The (00100)**'s.** The second SSBI contains three (00100)'s, which are obtained by contracting with $(\Gamma_{c})_{\delta}{}^{\gamma}$ and antisymmetrising in abc,

$$\begin{aligned} 0 = &-64D_{[a}A_{bc]} - D\Gamma_{[a}\tilde{t}_{bc]} - 2D\Gamma_{[ab}\tilde{t}_{c]} - D\Gamma_{abc}\tilde{t} + 4\tilde{S}_{[a}{}^{\alpha}\tilde{t}_{bc]\alpha} - 2\tilde{S}\Gamma_{[a}\tilde{t}_{bc]} - \\ &- 2\tilde{S}^{i}\Gamma_{[ab}\tilde{t}_{c]i} - 2\tilde{S}^{i}{}_{,[a}\Gamma_{b}\tilde{t}_{c]i} - 252\tilde{Z}_{[a}{}^{\alpha}\tilde{t}_{bc]\alpha} - 42\tilde{Z}\Gamma_{[a}\tilde{t}_{bc]} - 18\tilde{Z}'_{[a}{}^{\alpha}\tilde{t}_{bc]\alpha} - 35\tilde{Z}'\Gamma_{[a}\tilde{t}_{bc]} - \\ &- \frac{64}{3}A_{[a}{}^{i_1i_2i_3}A_{bc]i_1i_2i_3} + 64A_{[ab}{}^{i_1i_2}A'_{c]i_1i_2} + \frac{64}{3}A_{[a|i_1i_2i_3}B^{i_1i_2i_3}{}_{|b,c]} - \frac{64}{3}A'_{[a}{}^{i_1i_2i_3}A'_{bc]i_1i_2i_3} - \\ &- \frac{1}{9}\epsilon_{abci_1\dots i_8}A'^{i_1\dots i_4}A'^{i_5\dots i_8} - \frac{2}{45}\epsilon_{[ab|i_1\dots i_9}A'^{i_1\dots i_4}B^{i_5\dots i_9}{}_{,|c]}\,, \end{aligned} \tag{B.115}$$

$(\Gamma^{b}{}_{cd})_{\delta}{}^{\gamma}$ and antisymmetrising in acd,

$$\begin{aligned} 0 = &-256D_{[a}A'_{cd]} + 32D^{i}A_{iacd} + \frac{32}{3}D^{i}B_{acd,i} + 2D\Gamma_{[a}\tilde{t}_{cd]} - 5D\Gamma_{[ac}\tilde{t}_{d]} - 8D\Gamma_{acd}\tilde{t} - \\ &- 8\tilde{S}_{[a}{}^{\alpha}\tilde{t}_{cd]\alpha} + 4\tilde{S}\Gamma_{[a}\tilde{t}_{cd]} + 13\tilde{S}^{i}\Gamma_{[ac}\tilde{t}_{d]i} - 56\tilde{Z}_{[a}{}^{\alpha}\tilde{t}_{cd]\alpha} + 80\tilde{Z}_{[ac}{}^{i\alpha}\tilde{t}_{d]i\alpha} - 28\tilde{Z}\Gamma_{[a}\tilde{t}_{cd]} + \\ &+ 40\tilde{Z}_{[a}{}^{i}\Gamma_{c}\tilde{t}_{d]i} + 28\tilde{Z}^{i}\Gamma_{[ac}\tilde{t}_{d]i} + 20\tilde{Z}'_{[a}{}^{\alpha}\tilde{t}_{cd]\alpha} + 14\tilde{Z}'\Gamma_{[a}\tilde{t}_{cd]} + 8\tilde{Z}'_{[a}{}^{i}\Gamma_{c}\tilde{t}_{d]i} - 10\tilde{Z}'^{i}\Gamma_{[ac}\tilde{t}_{d]i} - \\ &- 384A^{i}A_{acdi} - 64A_{[a}{}^{i_1i_2}A_{cd]i_1i_2} + \frac{64}{3}A_{[a}{}^{i_1i_2i_3}A'_{cd]i_1i_2i_3} - \frac{128}{3}A_{[ac}{}^{i_1i_2i_3}A'_{d]i_1i_2i_3} + \\ &+ 192A'_{[a}{}^{i_1i_2}A'_{cd]i_1i_2} - \frac{448}{3}A'_{[a}{}^{i_1i_2i_3}B_{|i_1i_2i_3|c,d]} - \frac{4}{9}\epsilon_{acdi_1\dots i_8}A^{i_1\dots i_4}A'^{i_5\dots i_8} - \\ &- \frac{4}{45}\epsilon_{[ac|i_1\dots i_9|}A^{i_1\dots i_4}B^{i_5\dots i_9}{}_{,d]}\,, \end{aligned} \tag{B.116}$$

and with $(\Gamma^{ab}{}_{cde})_{\delta}{}^{\gamma}$,

$$0 = -448D^{i}A'_{icde} + 6D\Gamma_{[c}\tilde{t}_{de]} - 42D\Gamma_{[cd}\tilde{t}_{e]} + 56D\Gamma_{cde}\tilde{t} - 72\tilde{Z}_{[c}{}^{\alpha}\tilde{t}_{de]\alpha} - 96\tilde{Z}_{[cd}{}^{i\alpha}\tilde{t}_{e]i\alpha} -$$

$$
\begin{aligned}
&-32\tilde{Z}_{cde}{}^{i_1i_2\alpha}\tilde{t}_{i_1i_2\alpha}-60\tilde{Z}\Gamma_{[c}\tilde{t}_{de]}-144\tilde{Z}_{[c}{}^{i}\Gamma_{d}\tilde{t}_{e]i}-48\tilde{Z}_{[cd}{}^{i_1i_2}\Gamma_{e]}\tilde{t}_{i_1i_2}+72\tilde{Z}^{i}\Gamma_{[cd}\tilde{t}_{e]i}-\\
&-12\tilde{Z}^{i_1i_2}\Gamma_{cde}\tilde{t}_{i_1i_2}-48\tilde{Z}_{[c}{}^{i_1i_2}\Gamma_{de]}\tilde{t}_{i_1i_2}-12\tilde{Z}'_{[c}{}^{\alpha}\tilde{t}_{de]\alpha}+30\tilde{Z}'\Gamma_{[c}\tilde{t}_{de]}+48\tilde{Z}'_{[c}{}^{i}\Gamma_{d}\tilde{t}_{e]i}+\\
&+12\tilde{Z}'^{i}\Gamma_{[cd}\tilde{t}_{e]i}+4\tilde{Z}'^{i_1i_2}\Gamma_{cde}\tilde{t}_{i_1i_2}-384A_{[c}{}^{i_1i_2i_3}A_{de]i_1i_2i_3}+5376A^{i}A'_{cdei}+\\
&+2304A_{[c}{}^{i_1i_2}A'_{de]i_1i_2}+2304A_{[cd}{}^{i_1i_2}A'_{e]i_1i_2}+256A_{[c}{}^{i_1i_2i_3}B_{|i_1i_2i_3|d,e]}+\\
&+\frac{16}{9}\epsilon_{cdei_1\ldots i_8}A^{i_1\ldots i_4}A'^{i_5\ldots i_8}-\frac{16}{9}\epsilon_{cdei_1\ldots i_8}A'^{i_1\ldots i_4}A'^{i_5\ldots i_8}-\frac{32}{45}\epsilon_{cdei_1\ldots i_8}A'^{i_1i_2i_3j}B^{i_4\ldots i_8}{}_{,j}-\\
&-1280A'_{[c}{}^{i_1i_2i_3}A'_{de]i_1i_2i_3}+1152A'_{[cd}{}^{i_1i_2}B_{|i_1i_2|,e]}\,. \qquad \text{(B.117)}
\end{aligned}
$$

The (00002)'s. The second SSBI contains three (00002), which are obtained by contracting with $(\Gamma_{cde})_{\delta}{}^{\gamma}$ and antisymmetrising in $abcde$,

$$
\begin{aligned}
0=&\,64D_{[a}A_{bcde]}+D\Gamma_{[abc}\tilde{t}_{de]}+2D\Gamma_{[abcd}\tilde{t}_{e]}+D\Gamma_{abcde}\tilde{t}-26\tilde{S}_{[a}\Gamma_{bc}\tilde{t}_{de]}-2\tilde{S}\Gamma_{[abc}\tilde{t}_{de]}+\\
&+80\tilde{Z}_{[abc}{}^{\alpha}\tilde{t}_{de]\alpha}+60\tilde{Z}_{[ab}\Gamma_{c}\tilde{t}_{de]}+84\tilde{Z}_{[a}\Gamma_{bc}\tilde{t}_{de]}-14\tilde{Z}\Gamma_{[abc}\tilde{t}_{de]}+12\tilde{Z}'_{[ab}\Gamma_{c}\tilde{t}_{de]}-\\
&-30\tilde{Z}'_{[a}\Gamma_{bc}\tilde{t}_{de]}+7\tilde{Z}'\Gamma_{[abc}\tilde{t}_{de]}+384A_{[a}A_{bcde]}-384A_{[ab}{}^{i}A_{cde]i}-192A_{[abc}{}^{i_1i_2}A'_{de]i_1i_2}-\\
&-64A_{[ab}{}^{i_1i_2}A'_{cde]i_1i_2}+192A_{[abc}{}^{i}B_{de],i}-384A'_{[ab}{}^{i}A'_{cde]i}+128A'_{[ab}{}^{i_1i_2}B_{cde]i_1,i_2}-\\
&-\frac{8}{9}\epsilon_{[abcd|i_1\ldots i_7|}A^{i_1i_2i_3}{}_{e]}A'^{i_4\ldots i_7}-\frac{8}{45}\epsilon_{[abc|i_1\ldots i_8|}A^{i_1i_2i_3}{}_{d}B^{i_4\ldots i_8}{}_{,e]}-\\
&-\frac{8}{3}\epsilon_{abcdei_1\ldots i_6}A'^{i_1i_2}A'^{i_3\ldots i_6}-\frac{8}{9}\epsilon_{[abcd|i_1\ldots i_7|}A'^{i_1\ldots i_4}B^{i_5i_6i_7}{}_{,e]}\,, \qquad \text{(B.118)}
\end{aligned}
$$

$(\Gamma_{cdef})_{\delta}{}^{\gamma}$ and antisymmetrising in $abcdef$,

$$
\begin{aligned}
0=&\,64D_{[a}A_{bcdef]}+D\Gamma_{[abcd}\tilde{t}_{ef]}+2D\Gamma_{[abcde}\tilde{t}_{f]}+D\Gamma_{abcdef}\tilde{t}-26\tilde{S}_{[a}\Gamma_{bcd}\tilde{t}_{ef]}-\\
&-2\tilde{S}\Gamma_{[abcd}\tilde{t}_{ef]}+48\tilde{Z}_{[abcd}{}^{\alpha}\tilde{t}_{ef]\alpha}-64\tilde{Z}_{[abc}\Gamma_{d}\tilde{t}_{ef]}+120\tilde{Z}_{[ab}\Gamma_{cd}\tilde{t}_{ef]}-48\tilde{Z}_{[a}\Gamma_{bcd}\tilde{t}_{ef]}-\\
&-14\tilde{Z}\Gamma_{[abcd}\tilde{t}_{ef]}+24\tilde{Z}'_{[ab}\Gamma_{cd}\tilde{t}_{ef]}-24\tilde{Z}'_{[a}\Gamma_{bcd}\tilde{t}_{ef]}-\tilde{Z}'\Gamma_{[abcd}\tilde{t}_{ef]}-512A_{[ab}A_{cdef]}+\\
&+512A_{[abc}{}^{i}A'_{def]i}-128A_{[ab}{}^{i_1i_2}B_{cdef]i_1,i_2}+512A'_{[ab}A'_{cdef]}+\frac{512}{3}A'_{[abc}{}^{i}B_{def],i}-\\
&-\frac{32}{15}\epsilon_{[abcd|i_1\ldots i_7|}A^{i_1i_2}{}_{ef]}A'^{i_3\ldots i_7}-\frac{32}{9}\epsilon_{[abcd|i_1\ldots i_7|}A^{i_1i_2i_3}{}_{ef]}A'^{i_4\ldots i_7}-\\
&-\frac{64}{9}\epsilon_{abcdefi_1\ldots i_5}A'^{i_1i_2j}A'^{i_3i_4i_5}{}_{j}-\frac{32}{9}\epsilon_{[abcd|i_1\ldots i_7|}A'^{i_1\ldots i_4}B^{i_5i_6i_7}{}_{e,f]}\,, \qquad \text{(B.119)}
\end{aligned}
$$

and with $(\Gamma^{b}{}_{cdef})_{\delta}{}^{\gamma}$ and antisymmetrising in $acdef$,

$$
\begin{aligned}
0=&\,192D_{[a}A'_{cdef]}-\frac{32}{5}D^{i}B_{acdef,i}-\frac{4}{15}\epsilon_{acdef}{}^{i_1\ldots i_6}D_{i_1}A'_{i_2\ldots i_6}-4D\Gamma_{[acd}\tilde{t}_{ef]}+D\Gamma_{[acde}\tilde{t}_{f]}+\\
&+6D\Gamma_{acdef}\tilde{t}+33\tilde{S}_{[a}\Gamma_{cd}\tilde{t}_{ef]}+4\tilde{S}\Gamma_{[acd}\tilde{t}_{ef]}+\tilde{S}_{[a,}{}^{i}\Gamma_{cde}\tilde{t}_{f]i}+\tilde{S}^{i}\Gamma_{[acde}\tilde{t}_{f]i}+64\tilde{Z}_{[acd}{}^{\alpha}\tilde{t}_{ef]\alpha}-\\
&-32\tilde{Z}_{[acde}{}^{i\alpha}\tilde{t}_{f]\alpha}-144\tilde{Z}_{[ac}\Gamma_{d}\tilde{t}_{ef]}+64\tilde{Z}_{[acd}{}^{i}\Gamma_{e}\tilde{t}_{f]i}-144\tilde{Z}_{[a}\Gamma_{cd}\tilde{t}_{ef]}-96\tilde{Z}_{[ac}{}^{i}\Gamma_{de}\tilde{t}_{f]i}+\\
&+40\tilde{Z}\Gamma_{[acd}\tilde{t}_{ef]}+48\tilde{Z}_{[a}{}^{i}\Gamma_{cde}\tilde{t}_{f]i}-12\tilde{Z}^{i}\Gamma_{[acde}\tilde{t}_{f]i}+48\tilde{Z}'_{[ac}\Gamma_{d}\tilde{t}_{ef]}-24\tilde{Z}'_{[a}\Gamma_{cd}\tilde{t}_{ef]}-\\
&-20\tilde{Z}'\Gamma_{[acd}\tilde{t}_{ef]}-16\tilde{Z}'_{[a}{}^{i}\Gamma_{cde}\tilde{t}_{f]i}-2\tilde{Z}'^{i}\Gamma_{[acde}\tilde{t}_{f]i}+1152A_{[a}A'_{cdef]}-1024A_{[ac}{}^{i}A'_{def]i}-\\
&-128A_{[ac}{}^{i_1i_2}A_{def]i_1i_2}+1536A_{[acd}{}^{i}A'_{ef]i}+256A'_{[ac}{}^{i_1i_2}A'_{def]i_1i_2}+896A'_{[acd}{}^{i}B_{ef],i}-\\
&-\frac{8}{3}\epsilon_{acdefi_1\ldots i_6}A^{i_1i_2}A'^{i_3\ldots i_6}-\frac{16}{3}\epsilon_{acdefi_1\ldots i_6}A^{i_1\ldots i_4}A'^{i_5i_6}-
\end{aligned}
$$

$$-\frac{16}{9}\epsilon_{[acde|i_1\dots i_7|}A^{i_1\dots i_4}B^{i_5i_6i_7}{}_{,f]} - \frac{112}{45}\epsilon_{[acd|i_1\dots i_8|}A'^{i_1i_2i_3}{}_e B^{i_4\dots i_8}{}_{,f]}\,, \tag{B.120}$$

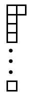
where we have used (B.66) and the following identities

$$\begin{aligned}
\epsilon_{abcdei_1\dots i_6}A^{i_1i_2i_3j}A'^{i_4i_5i_6}{}_j &= -\frac{5}{4}\epsilon_{[abcd|i_1\dots i_7|}A^{i_1\dots i_4}A'^{i_5i_6i_7}{}_{e]}\,,\\
\epsilon_{abcdei_1\dots i_6}A^{i_1i_2i_3j}A^{i_4i_5i_6}{}_j &= -\frac{5}{4}\epsilon_{[abcd|i_1\dots i_7|}A^{i_1\dots i_4}A^{i_5i_6i_7}{}_{e]} = 0\,,\\
\epsilon_{abcdei_1\dots i_6}A'^{i_1i_2i_3j}B^{i_4i_5i_6}{}_{,j} &= -\frac{5}{4}\epsilon_{[abcd|i_1\dots i_7|}A'^{i_1\dots i_4}B^{i_5i_6i_7}{}_{,e]}\,,\\
\epsilon_{[abcd|i_1\dots i_7|}A^{i_1i_2i_3}{}_{ef]}A'^{i_4\dots i_7} &= \frac{4}{5}\epsilon_{[abcde|i_1\dots i_6|}A^{i_1i_2i_3j}{}_{f]}A'^{i_4i_5i_6}{}_j\,,\\
\epsilon_{[abcd|i_1\dots i_7|}A^{i_1i_2}{}_{ef]}A'^{i_3\dots i_7} &= \epsilon_{[abcde|i_1\dots i_6|}A^{i_1i_2j}{}_{f]}A'^{i_3\dots i_6}{}_j\,,\\
\epsilon_{[abcd|i_1\dots i_7|}A'^{i_1\dots i_4}B^{i_5i_6i_7}{}_{e,f]} &= -\frac{4}{5}\epsilon_{[abcde|i_1\dots i_6}A'^{i_1i_2i_3j}B^{i_4i_5i_6}{}_{j,|f]}\,,\\
\epsilon_{abcdefi_1\dots i_5}A'^{i_1i_2j}A'^{i_3i_4i_5}{}_j &= -\frac{3}{2}\epsilon_{[abcde|i_1\dots i_6|}A'^{i_1i_2}{}_{f]}A'^{i_3\dots i_6}\\
&= 2\epsilon_{[abcde|i_1\dots i_6|}A'^{i_1i_2i_3}A'^{i_4i_5i_6}{}_{f]}
\end{aligned} \tag{B.121}$$

and

$$\begin{aligned}
\epsilon_{[abcd|i_1\dots i_7|}A'^{i_1\dots i_4}A'^{i_5i_6i_7}{}_{ef]} &= -\frac{4}{5}\epsilon_{[abcde|i_1\dots i_6}A'^{i_1i_2i_3j}A'^{i_4i_5i_6}{}_{j|f]}\\
\epsilon_{[abcde|i_1\dots i_6}A'^{i_1i_2i_3j}A'^{i_4i_5i_6}{}_{j|f]} &= -\frac{1}{2}\epsilon_{abcdefi_1\dots i_5}A'^{i_1i_2jk}A'^{i_3i_4i_5}{}_{jk}\\
\epsilon_{[abcd|i_1\dots i_7|}A'^{i_1i_2}{}_{ef]}A'^{i_3\dots i_7} &= \epsilon_{[abcde|i_1\dots i_6|}A'^{i_1i_2j}{}_{f]}A'^{i_3\dots i_6}{}_j\\
\epsilon_{[abcde|i_1\dots i_6|}A'^{i_1i_2j}{}_{f]}A'^{i_3\dots i_6}{}_j &= \frac{2}{3}\epsilon_{abcdefi_1\dots i_5}A'^{i_1i_2jk}A'^{i_3i_4i_5}{}_{jk}\,.
\end{aligned} \tag{B.122}$$

C. Decomposition of tensor-spinors

Decomposition of tensor-spinors of the types  and .

1. Consider a general (= reducible) rank-n antisymmetric tensor-spinor in D dimensions, $V^\alpha_{a_1\dots a_n}$. We want to decompose it into irreducible (Γ-traceless) representations. An irreducible rank-n tensor spinor is obtained from a reducible one as

$$V'{}_{a_1\dots a_n} = \sum_{p=0}^{n} N_{n,p}\Gamma_{[a_1\dots a_p}\Gamma^{b_1\dots b_p}V_{|b_1\dots b_p|a_{p+1}\dots a_n]}\,, \tag{C.1}$$

where $N_{n,p} = \frac{(-1)^{\frac{p(p+1)}{2}}}{p!}\frac{\binom{n}{p}}{\binom{D-2n+p+1}{p}}$.

If we define the expansion of V in irreducible representations as

$$V_{a_1\dots a_n} = \sum_{p=0}^{n}\binom{n}{p}\Gamma_{[a_1\dots a_p}\tilde{V}_{a_{p+1}\dots a_n]}\,, \tag{C.2}$$

the Γ-traces of V are

$$v_{a_{p+1}\dots a_n} \equiv \frac{1}{p!}\Gamma^{a_1\dots a_p}V_{a_1\dots a_p a_{p+1}\dots a_n} \tag{C.3}$$
$$= (-1)^{\frac{p(p-1)}{2}} \sum_{r=0}^{n-p} \binom{n-p}{r}\binom{D-2n+2p+r}{p} \Gamma_{[a_{p+1}\dots a_{p+r}} \tilde{V}_{a_{p+r+1}\dots a_n]} \,.$$

Subtracting the Γ-trace leaves only the first term in the sum:

$$v'_{a_{p+1}\dots a_n} = (-1)^{\frac{p(p-1)}{2}} \binom{D-2n+2p}{p} \tilde{V}_{a_{p+1}\dots a_n} \,, \tag{C.4}$$

or, explicitly, using (C.1):

$$\tilde{V}_{a_{p+1}\dots a_n} = \frac{(-1)^p}{\binom{D-2n+2p}{p}} \sum_{r=p}^{n} \frac{(-1)^{\frac{r(r+1)}{2}}}{r!} \frac{\binom{r}{p}\binom{n-p}{r-p}}{\binom{D-2n+p+r+1}{r-p}} \Gamma_{[a_{p+1}\dots a_r} \Gamma^{b_1\dots b_r} V_{|b_1\dots b_r|a_{r+1}\dots a_n]} \,, \tag{C.5}$$

which of course coincides with (C.1) for $p=0$.

2. Consider the tensor product of an irreducible hook tensor, i.e., a tensor of the type

$$U_{a_1\dots a_n,a} : \qquad U_{[a_1\dots a_n,a]} = 0, \quad U_{a_1\dots a_{n-1}a}{}^{a} = 0 \,, \tag{C.6}$$

with a spinor. The Γ-traceless part is

$$\begin{aligned} U'_{a_1\dots a_n,a} = & \sum_{p=1}^{n} k_{n,p} \Gamma_{[a_1\dots a_p} \Gamma^{b_1\dots b_p} U_{|b_1\dots b_p|a_{p+1}\dots a_n],a} + \\ & + \sum_{p=1}^{n} l_{n,p} \Gamma_{a[a_1\dots a_{p-1}} \Gamma^{b_1\dots b_p} U_{|b_1\dots b_p|a_p\dots a_{n-1},a_n]} + \\ & + \sum_{p=2}^{n} m_{n,p} \eta_{a[a_1} \Gamma_{a_2\dots a_{p-1}} \Gamma^{b_1\dots b_p} U_{|b_1\dots b_p|a_p\dots a_{n-1},a_n]} - \\ & - [a_1 \dots a_n a] \,, \end{aligned} \tag{C.7}$$

where

$$\begin{aligned} k_{n,p} &= N_{n,p} \,, \\ l_{n,p} &= (-1)^{n+1} \frac{(D-n+1)(D-2n+1)-(n+1)p}{(D+2)(D-n+2)} N_{n,p} \,, \\ m_{n,p} &= (-1)^n \frac{(D-2n+p+1)(n+1)(p-1)}{(D+2)(D-n+2)} N_{n,p} \end{aligned} \tag{C.8}$$

(at $p=n$, only the combination $\frac{1}{n+1}(nk_{n,n} + (-1)^{n+1} l_{n,n}) = \frac{(D+1)(D-n+1)}{(D+2)(D-n+2)} N_{n,n}$ enters).

The vanishing of the completely antisymmetric part implies that Γ-traces only have to be taken on $a_1 \dots a_n$, and Γ-tracelessness in these indices implies Γ-tracelessness in a. The tracelessness wrt the vector indices survives after a multiple Γ-trace, but the vanishing of the antisymmetrised tensor does not, so in order to get an irreducible representation out of a Γ-trace on U one has to divide it into an antisymmetric part and an irreducible hook, and subtract the Γ-traces of both. Explicitly:

$$u_{a_{p+1}\ldots a_n,a} \equiv \frac{1}{p!}\Gamma^{a_1\ldots a_p}U_{a_1\ldots a_n,a} \tag{C.9}$$

splits into

$$u_{a_{p+1}\ldots a_n,a} = v_{a_{p+1}\ldots a_n a} + w_{a_{p+1}\ldots a_n,a}\,, \tag{C.10}$$

where

$$\begin{aligned} v_{a_{p+1}\ldots a_n a} &\equiv u_{[a_{p+1}\ldots a_n,a]}\,,\\ w_{a_{p+1}\ldots a_n,a} &\equiv u_{a_{p+1}\ldots a_n,a} - u_{[a_{p+1}\ldots a_n,a]}\\ &= \frac{n-p}{n-p+1}\left(u_{a_{p+1}\ldots a_n,a} + (-1)^{n-p+1}u_{a[a_{p+1}\ldots a_{n-1},a_n]}\right)\,. \end{aligned} \tag{C.11}$$

The tensor $u'_{a_{p+1}\ldots a_n,a}$ is defined using the subtraction of Γ-traces according to (C.1) and (C.7), and consists of two irreducible tensors v' and w'. One obtains:

$$v'_{a_{p+1}\ldots a_n a} = \sum_{r=p}^{n}(-1)^{\frac{p(p+1)}{2}+\frac{r(r+1)}{2}}\frac{1}{r!}\frac{\binom{r-1}{p-1}\binom{n-p}{r-p}}{\binom{D-2n+p+r-1}{r-p}}\Gamma_{[a_{p+1}\ldots a_r}\Gamma^{b_1\ldots b_r}U_{|b_1\ldots b_r|a_{r+1}\ldots a_n,a]} \tag{C.12}$$

On the other hand, an expansion of U in irreducible tensors is defined by

$$\begin{aligned} U_{a_1\ldots a_n,a} = &\sum_{p=0}^{n-1}\binom{n}{p}\Gamma_{[a_1\ldots a_p}\tilde{W}_{a_{p+1}\ldots a_n],a}+\\ &+\sum_{p=1}^{n}\binom{n}{p}\Bigg(\Gamma_{a[a_1\ldots a_{p-1}}\tilde{V}_{a_p\ldots a_n]} + (-1)^{n-1}\Gamma_{[a_1\ldots a_p}\tilde{V}_{a_{p+1}\ldots a_n]a}+\\ &\qquad+\frac{(n+1)(p-1)}{D-n+1}\eta_{a[a_1}\Gamma_{a_2\ldots a_{p-1}}\tilde{V}_{a_p\ldots a_n]}\Bigg)\,. \end{aligned} \tag{C.13}$$

Performing the Γ-traces on this expansion yields (the second eq. directly from (C.4), the first after some computing)

$$\begin{aligned} v'_{a_{p+1}\ldots a_n a} &= (-1)^{n+1+\frac{p(p-1)}{2}}\frac{(D+2)(D-n+p)}{(D-n+1)(D-2n+p-1)}\binom{D-2n+2p-2}{p}\tilde{V}_{a_{p+1}\ldots a_n a}\,,\\ w'_{a_{p+1}\ldots a_n,a} &= (-1)^{\frac{p(p-1)}{2}}\binom{D-2n+2p}{p}\tilde{W}_{a_{p+1}\ldots a_n,a}\,. \end{aligned} \tag{C.14}$$

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